

Next-to-leading order thermal spectral functions in the perturbative domain

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Abstract

Motivated by applications in thermal QCD and cosmology, we elaborate on a general method for computing next-to-leading order spectral functions for composite operators at vanishing spatial momentum, accounting for real, virtual as well as thermal corrections. As an example, we compute these functions (together with the corresponding imaginary-time correlators which can be compared with lattice simulations) for scalar and pseudoscalar densities in pure Yang-Mills theory. Our results may turn out to be helpful in non-perturbative estimates of the corresponding transport coefficients, which are the bulk viscosity in the scalar channel and the rate of anomalous chirality violation in the pseudoscalar channel. We also mention links to cosmology, although the most useful results in that context may come from a future generalization of our methods to other correlators.

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1. Introduction

A “spectral function”, ρ , which has a Minkowskian four-momentum, \mathcal{P} , as its argument, can formally be defined as a Fourier transform of the thermal expectation value ($\langle \dots \rangle_T \equiv \frac{1}{\mathcal{Z}} \text{Tr} [e^{-\beta \hat{H}} (\dots)]$) of the commutator of a given gauge-invariant local operator:

$$\rho(\mathcal{P}) \equiv \int_{-\infty}^{\infty} dt \int_{\mathbf{x}} e^{i\mathcal{P} \cdot \mathcal{X}} \left\langle \frac{1}{2} [\hat{O}(t, \mathbf{x}), \hat{O}(0, \mathbf{0})] \right\rangle_T, \quad (1.1)$$

where $\mathcal{X} \equiv (t, \mathbf{x})$, the operator is defined in the Heisenberg picture, and $\mathcal{P} \cdot \mathcal{X} = \omega t - \mathbf{p} \cdot \mathbf{x}$. All other orderings (retarded, time-ordered, Wightman, etc.) can be expressed in terms of the spectral function [1, 2]. Therefore various spectral functions play an important role in theoretical analyses of thermal systems. In particular, in the “linear response” regime, the production rate of a weakly-coupled particle species from the medium is proportional to the spectral function of the operator that the particle in question couples to [1, 2].

Because of the fundamental role of spectral functions, they make an appearance in many different areas of many-body physics; in the following, selected examples having a connection to our present study are outlined in some more detail.

QCD at high temperatures

Various “transport coefficients”, which characterize the relaxation of local excesses in conserved currents (momentum; number densities; electromagnetic current) to their equilibrium values, can be viewed as the “low-energy constants” of an interacting hot plasma (cf. e.g. ref. [3]). They can be defined through the limit $\lim_{\omega \rightarrow 0^+} \rho(\omega, \mathbf{0})/\omega$ of an appropriate spectral function. In recent years considerable efforts have been devoted to determining these quantities at temperatures around a few hundred MeV, relevant for the current experimental heavy ion collision program; an extensive review, with a particular perspective on non-perturbative lattice measurements, can be found in ref. [4].

Unfortunately, as becomes clear from ref. [4], a reliable non-perturbative determination of transport coefficients represents a formidable challenge, even in the limit of high temperatures where it can be argued that pure Yang-Mills theory should already yield a qualitatively correct description. The problem is that direct lattice measurements concern Euclidean correlators, whereas transport coefficients are related to the longest Minkowskian time scales characterizing the system. Even though the two situations can in principle be related to each other through an analytic continuation [5], a necessary requirement appears to be the subtraction of short-distance divergences [6], and even then the problem remains numerically ill-posed (see e.g. ref. [4] and references therein). It therefore seems pertinent to obtain as much information as possible by analytic means, in order to carry out a subtraction leaving over a simple function containing (mostly) only infrared physics. Thanks to asymptotic

freedom, it is possible to address analytically not only the temperature-independent short-distance divergences but also, at high enough temperatures, thermal modifications originating from the large-frequency range $\omega \sim \pi T$, and this is one of the goals of the present work.

QCD at low temperatures

At low temperatures, where chiral symmetry is spontaneously broken, numerical simulations are particularly demanding; on the other hand, chiral perturbation theory [7], if pursued to a sufficient order, may provide for a reasonably accurate description of this regime. Since mesons are bosons, the analytic computations encountered are technically not unlike those at very high temperatures in pure Yang-Mills theory, and many of the same methods may turn out to be useful. This then serves as an additional motivation for developing analytic methods. Recent computations of various transport coefficients within chiral perturbation theory (or extensions thereof) can be found in refs. [8]–[12]; we envisage that our techniques and generalizations thereof may allow to address some of the open issues outlined in ref. [8].

Cosmology

In cosmology, particle production rates are relevant for certain Dark Matter scenarios, as well as for leptogenesis computations in which a non-thermal distribution of unstable particles serves as an intermediate stage. Typically, one is then interested in the whole non-equilibrium spectrum of the particles produced, proportional to a spectral function evaluated at $\mathcal{P} = (E_p, \mathbf{p})$, where \mathbf{p} is the spatial momentum and $E_p \equiv \sqrt{p^2 + M^2}$ (cf. e.g. ref. [13]). However, if the particles produced are non-relativistic, with a mass $M \gg T$, then it can be argued that the spectral function at $\mathcal{P} = (M, \mathbf{0})$ already yields some of the information. An example of a recent analysis in this spirit, in which a subset of the 2-loop topologies studied in the present paper were computed to leading non-trivial order in an expansion in T^2/M^2 (we refer to this approximation as the “OPE-limit”), can be found in ref. [14]. Since our techniques work beyond the OPE-limit, we expect them to allow for an improvement of such computations. A few other cosmological applications are suggested in the Conclusions.

Previous work

The very first spectral function computed at next-to-leading order (NLO) in the domain $\omega \sim \pi T$, $\mathbf{p} = \mathbf{0}$ was that related to the electromagnetic current associated with massless quarks [15]–[17]. These results have subsequently been generalized to the case of a finite (heavy) quark mass [18], and another spectral function related to this case, correlating Lorentz forces felt by the nearly static heavy quarks, was derived at NLO with similar methods [19]. A general analysis of many spectral functions in the domain $\omega \gg \pi T$, which can be systematized through Operator Product Expansion (OPE) techniques, was carried out in ref. [20]. The

current study aims to generalize the approach presented in refs. [18, 19]. Note that there is a huge body of literature concerning the domain $\omega \lesssim gT$, $\mathbf{p} = \mathbf{0}$ (here $g = \sqrt{4\pi\alpha_s}$), in which perturbation theory breaks down; upon this we touch only briefly in secs. 5.2, 5.3.

Outline of this paper

The observables that we are specifically concerned with are the 2-point correlators of the scalar (“ FF ”) and pseudoscalar (“ $F\tilde{F}$ ”) densities in pure $SU(N_c)$ Yang-Mills theory. These quantities have recently been studied particularly by Meyer, who has addressed both the Minkowskian and Euclidean domains [21]–[24]. Noteworthy is also a general discussion concerning the shape of the scalar channel spectral function at small frequencies [25] as well as the mentioned OPE representation of its ultraviolet asymptotics [20]. In addition, sum rules have been analyzed [26], improving on earlier findings [27, 28], and further insight has been sought from AdS/CFT inspired models [23, 29, 30, 31]. The corresponding transport coefficients (this yields the bulk viscosity for FF , cf. e.g. ref. [32], and the rate of anomalous chirality changing transitions for $F\tilde{F}$, cf. ref. [33]), together with possible applications, have been the subject of much further work; we restrict here to mentioning determinations of the transport coefficients in the weak-coupling limit [34, 35].

The current work is a continuation of two recent papers which also addressed the same general problems. In ref. [36], the scalar and pseudoscalar correlators were studied within the OPE framework, refining the previous determination of certain Wilson coefficients and also correcting results concerning the pseudoscalar channel [23]. In ref. [37], which went beyond the OPE-limit, the full $|\mathbf{x}|$ -dependences of the time-averaged Euclidean correlators were determined, and compared with related lattice measurements carried out in ref. [23]. One specific purpose of the present paper is to “complement” ref. [37], by generalizing the results beyond the OPE-limit also in the Minkowskian frequency domain.

As already mentioned our current analysis borrows techniques from refs. [18, 19] but goes beyond these works in its generality. We hope that we can thereby pave the way for a number of further generalizations, such as the inclusion of the Lorentz non-invariant structures appearing e.g. in the “shear” channel of the energy-momentum tensor; of a non-zero spatial momentum, relevant for relativistic particle production rates in cosmology; and of effects from a non-trivial mass spectrum, relevant e.g. in chiral perturbation theory applications.

The paper is organized as follows. The observables are defined in sec. 2, the general method for their determination is discussed in sec. 3, and the basic results are presented in sec. 4. The results are analyzed in some detail in sec. 5, which also contains numerical evaluations. We conclude and offer an outlook in sec. 6. Three appendices, A, B and C, contain a detailed investigation of one of the sum-integrals, concise results for the others, and an account of a resummation relevant for $g^2T/\pi \ll \omega \ll \pi T$, respectively.

2. Specific setup

Our basic notation follows ref. [36], so we discuss the specific setup only briefly. We consider pure $SU(N_c)$ Yang-Mills theory (with $N_c = 3$ in numerical estimates), dimensionally regularized by analytically continuing to $D = 4 - 2\epsilon$ space-time dimensions (the Euclidean action reads $S_E = \int_0^\beta d\tau \int d^{3-2\epsilon} \mathbf{x} \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a$). The operators considered are

$$\theta \equiv c_\theta g_B^2 F_{\mu\nu}^a F_{\mu\nu}^a, \quad \chi \equiv c_\chi \epsilon_{\mu\nu\rho\sigma} g_B^2 F_{\mu\nu}^a F_{\rho\sigma}^a, \quad (2.1)$$

where g_B^2 is the bare gauge coupling squared; these operators require no renormalization at the order of our computation. We normally leave the coefficients c_θ, c_χ unspecified, but note that often the values $c_\theta \approx -\frac{b_0}{2} - \frac{b_1 g^2}{4}$, $c_\chi \equiv \frac{1}{64\pi^2}$ are chosen, where $b_0 \equiv \frac{11N_c}{3(4\pi)^2}$ and $b_1 \equiv \frac{34N_c^2}{3(4\pi)^4}$. The bare gauge coupling squared can be expanded in terms of the renormalized one, g^2 , as

$$g_B^2 = g^2 \mu^{2\epsilon} \left[1 - \frac{g^2 b_0}{\epsilon} + \dots \right], \quad (2.2)$$

where μ denotes a scale parameter. The $\overline{\text{MS}}$ scheme renormalization scale is denoted by $\bar{\mu}^2 \equiv 4\pi\mu^2 e^{-\gamma_E}$. In order to avoid unnecessary clutter all appearances of $\mu^{2\epsilon}$, which play no role in the final renormalized results, will be suppressed.

The observables considered are related to the 2-point correlators of θ and χ . We define the quantities ($X \equiv (\tau, \mathbf{x})$; $P \equiv (p_n, \mathbf{p})$; $p_n \equiv 2\pi T n$, $n \in \mathbb{Z}$)

$$G_\theta(X) \equiv \langle \theta(X) \theta(0) \rangle_T, \quad G_\chi(X) \equiv \langle \chi(X) \chi(0) \rangle_T, \quad (2.3)$$

as well as the corresponding Fourier transforms

$$\tilde{G}_\theta(P) \equiv \int_X e^{-iP \cdot X} G_\theta(X), \quad \tilde{G}_\chi(P) \equiv \int_X e^{-iP \cdot X} G_\chi(X), \quad (2.4)$$

where $\int_X \equiv \int_0^\beta d\tau \int_{\mathbf{x}}$. No hats appear above the operators because Euclidean correlators can be evaluated with standard path integral techniques. It can be shown (cf. e.g. refs. [1, 2]) that the corresponding spectral functions, defined as in eq. (1.1) with $\hat{O} \rightarrow \hat{\theta}, \hat{\chi}$, are then obtained (at vanishing spatial momentum) from

$$\rho(\omega) = \text{Im} \left[\tilde{G}(P) \right]_{P \rightarrow (-i[\omega+i0^+], \mathbf{0})}. \quad (2.5)$$

It is these quantities (denoted by ρ_θ, ρ_χ) that we are mostly interested in.

3. General method

3.1. Wick contractions

Having defined the correlators, the first step in their determination is to carry out the Wick contractions (for our specific example the corresponding graphs are shown in fig. 1), and to

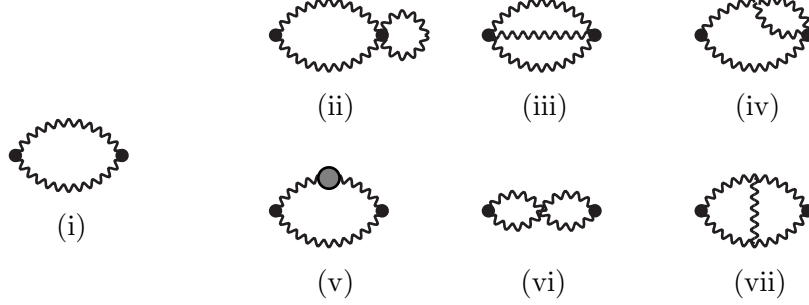


Figure 1: The graphs contributing to the 2-point correlators up to 2-loop order. The wiggly lines denote gluons; the small dots the operators θ or χ (cf. eq. (2.1)); and the grey blob the 1-loop gauge field self-energy. “Disconnected” contractions only arise at higher orders.

reduce the functions $\tilde{G}_\theta(P)$, $\tilde{G}_\chi(P)$ to sums over various types of integrals. We strongly advocate taking this step in Euclidean signature, whereby the Feynman rules are elementary (no excessive i ’s or doublings of degrees of freedom appear), and by using bare parameters, whereby counterterm vertices are avoided. An additional strength of the imaginary-time formalism is that in terms of physical processes, all “real” and “virtual” corrections related to each other are automatically captured by a single Matsubara sum-integral. Before carrying out the contractions, it is helpful to make use of translational invariance in order to write

$$\begin{aligned}\tilde{G}_\theta(P) &= \int_X e^{-iP \cdot X} \langle \theta(X) \theta(0) \rangle_T = \int_X e^{-iP \cdot (X-Y)} \langle \theta(X) \theta(Y) \rangle_T \\ &= \frac{1}{\int_Y} \left\langle \int_X e^{-iP \cdot X} \theta(X) \int_Y e^{iP \cdot Y} \theta(Y) \right\rangle_T,\end{aligned}\tag{3.1}$$

so that the full analysis can be carried out in momentum space (the factor $\int_Y = \beta V = \beta \delta_{p_n=0} (2\pi)^d \delta^{(d)}(\mathbf{p} = \mathbf{0})$, with $d = D - 1$, cancels out because of momentum conservation).

3.2. Scalarization

In the next step, the goal is to “scalarize” the sum-integrals, i.e. to turn them into ones that also appear in scalar field theories. This can be achieved by contracting Lorentz indices (using $\delta_{\mu\mu} = D$) and carrying out Dirac traces (with rules following from $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \mathbb{1}_{4 \times 4}$; in the case of chiral fermions the usual issues with γ_5 need to be faced and we have nothing to add on this topic). Subsequently, substitutions of sum-integration variables, completions of squares (e.g. $Q \cdot R = \frac{1}{2} [Q^2 + R^2 - (Q - R)^2]$), and identities following from partial integrations with respect to spatial momenta, can be employed in order to remove as many scalar products as possible from the numerators. The goal is to express the result in terms of a minimal number

of independent “master” sum-integrals. (If the set is minimal indeed, then all dependence on the gauge fixing parameter must have cancelled after this step.)

For the specific correlators $\tilde{G}_\theta(P)$ and $\tilde{G}_\chi(P)$, the reduction to master sum-integrals was carried out already in ref. [36]. Re-expressing a sum-integral “ \mathcal{I}_i ” of ref. [36] in terms of “ \mathcal{I}_i ” (cf. eq. (A.14) there) the results from eqs. (3.1), (3.2) of ref. [36] become ($d_A \equiv N_c^2 - 1$)

$$\begin{aligned} \frac{\tilde{G}_\theta(P)}{4d_A c_\theta^2} &= g_B^4 (D-2) \left[-\mathcal{J}_a + \frac{1}{2} \mathcal{J}_b \right] \\ &+ g_B^6 N_c \left\{ 2(D-2) \left[-(D-2)\mathcal{I}_a + (D-4)\mathcal{I}_b \right] + (D-2)^2 \left[\mathcal{I}_c - \mathcal{I}_d \right] \right. \\ &\quad \left. + \frac{34-13D}{3} \mathcal{I}_f - \frac{(D-4)^2}{2} \mathcal{I}_g + (D-2) \left[-\mathcal{I}_e + 3\mathcal{I}_h + 2\mathcal{I}_i - \mathcal{I}_j \right] \right\}, \end{aligned} \quad (3.2)$$

$$\begin{aligned} \frac{\tilde{G}_\chi(P)}{-16d_A c_\chi^2 (D-3)} &= g_B^4 (D-2) \left[-\mathcal{J}_a + \frac{1}{2} \mathcal{J}_b \right] \\ &+ g_B^6 N_c \left\{ 2(D-2)(D-4)\mathcal{I}_b + (D-2)^2 \left[\mathcal{I}_c - \mathcal{I}_d \right] \right. \\ &\quad \left. - \frac{2D^2 - 11D + 30}{3} \mathcal{I}_f - 2(D-4)\mathcal{I}_g + (D-2) \left[-\mathcal{I}_e + 3\mathcal{I}_h + 2\mathcal{I}_i - \mathcal{I}_j \right] \right\}. \end{aligned} \quad (3.3)$$

Here, for brevity, structures containing $\mathbb{F}_Q 1$, which vanishes exactly in dimensional regularization, have been omitted ($\mathbb{F}_Q \equiv T \sum_{q_n} \int_{\mathbf{q}}$). The master sum-integrals are defined as

$$\mathcal{J}_a(P) \equiv \oint_Q \frac{P^2}{Q^2}, \quad (3.4)$$

$$\mathcal{J}_b(P) \equiv \oint_Q \frac{P^4}{Q^2(Q-P)^2}, \quad (3.5)$$

$$\mathcal{I}_a(P) \equiv \oint_{Q,R} \frac{1}{Q^2 R^2}, \quad (3.6)$$

$$\mathcal{I}_b(P) \equiv \oint_{Q,R} \frac{P^2}{Q^2 R^2 (R-P)^2}, \quad (3.7)$$

$$\mathcal{I}_c(P) \equiv \oint_{Q,R} \frac{P^2}{Q^2 R^4}, \quad (3.8)$$

$$\mathcal{I}_d(P) \equiv \oint_{Q,R} \frac{P^4}{Q^2 R^4 (R-P)^2}, \quad (3.9)$$

$$\mathcal{I}_e(P) \equiv \oint_{Q,R} \frac{P^2}{Q^2 R^2 (Q-R)^2}, \quad (3.10)$$

$$\mathcal{I}_f(P) \equiv \oint_{Q,R} \frac{P^2}{Q^2 (Q-R)^2 (R-P)^2}, \quad (3.11)$$

$$\mathcal{I}_g(P) \equiv \oint_{Q,R} \frac{P^4}{Q^2(Q-P)^2 R^2(R-P)^2}, \quad (3.12)$$

$$\mathcal{I}_h(P) \equiv \oint_{Q,R} \frac{P^4}{Q^2 R^2(Q-R)^2(R-P)^2}, \quad (3.13)$$

$$\mathcal{I}_i(P) \equiv \oint_{Q,R} \frac{4(Q \cdot P)^2}{Q^2 R^2(Q-R)^2(R-P)^2}, \quad (3.14)$$

$$\mathcal{I}_j(P) \equiv \oint_{Q,R} \frac{P^6}{Q^2 R^2(Q-R)^2(Q-P)^2(R-P)^2}. \quad (3.15)$$

3.3. Matsubara sums and discontinuities

The master sum-integrals defined in eqs. (3.4)–(3.15) contain a 1- or 2-fold Matsubara sum. Both sums can be carried out explicitly. In the literature various recipes can be found for this, most notably “cutting rules”; we have carried out the sums with the method described in some detail in appendix A.1 of ref. [18] and illustrated with simple examples also in eqs. (C.13)–(C.15) and (C.29)–(C.30) below. Partial fractioning the result in the variable p_n , there may be polynomial terms, as well as fractions of the type

$$\frac{1}{ip_n + \sum_k \sigma_k E_k}, \quad (3.16)$$

where $\sigma_k = \pm 1$ and $E_k \in \{E_{\mathbf{q}}, E_{\mathbf{q}-\mathbf{p}}, E_{\mathbf{r}}, E_{\mathbf{r}-\mathbf{p}}, E_{\mathbf{q}-\mathbf{r}}\}$, $E_{\mathbf{q}} \equiv |\mathbf{q}|$. According to eq. (2.5), the corresponding spectral function follows from

$$\text{Im} \left[\frac{1}{\omega + i0^+ + \sum_k \sigma_k E_k} \right] = -\pi \delta(\omega + \sum_k \sigma_k E_k). \quad (3.17)$$

The spectral functions corresponding to all the master sum-integrals after this step (omitting terms $\propto \omega^n \delta(\omega)$ after setting $\mathbf{p} \rightarrow \mathbf{0}$, cf. sec. 5.6) are listed in appendices A and B.

3.4. Spatial integrals

The “hard work” of the problem, where little automatization seems possible, is to carry out the remaining spatial integrals, constrained by the Dirac- δ ’s. There are two classes of integrals, referred to in the literature as “virtual” and “real” corrections; we often refer to them as “factorized” and “phase space” integrals, respectively. Within the former class, some integrals are ultraviolet divergent, necessitating a careful handling by e.g. dimensional regularization. All classes of integrals are described in detail in appendix A for the specific spectral function denoted by $\rho_{\mathcal{I}_j}$, corresponding to the sum-integral defined in eq. (3.15), and more concise results are collected in appendix B for all the other cases.

3.5. Renormalization

After all master spectral functions are known, these are inserted into (the imaginary parts of) eqs. (3.2), (3.3). Simultaneously, $D = 4 - 2\epsilon$ is inserted into the coefficients, and the bare gauge coupling is re-expanded in terms of the renormalized one according to eq. (2.2). If the computation has been carried out correctly, all $1/\epsilon$ -divergences must cancel at this stage, and we can subsequently set $\epsilon \rightarrow 0$.

Let us illustrate the procedure with the example at hand. A number of the sum-integrals defined have no imaginary part ($\mathcal{J}_a, \mathcal{I}_a, \mathcal{I}_c, \mathcal{I}_e$), and therefore lead to a vanishing spectral function. Among the non-zero spectral functions, only three have an $1/\epsilon$ -divergence ($\rho_{\mathcal{I}_g}, \rho_{\mathcal{I}_h}, \rho_{\mathcal{I}_i}$). Taking these facts into account, eqs. (2.5), (3.2), (3.3) can be converted into

$$\begin{aligned} \frac{\rho_\theta(\omega)}{4d_A c_\theta^2} &= (1 - \epsilon) \left\{ g_B^4 \rho_{\mathcal{J}_b}(\omega) + 2g_B^6 N_c [3\rho_{\mathcal{I}_h}(\omega) + 2\rho_{\mathcal{I}_i}(\omega)] \right\} \\ &\quad - 2g_B^6 N_c [2\rho_{\mathcal{I}_d}(\omega) + 3\rho_{\mathcal{I}_f}(\omega) + \rho_{\mathcal{I}_j}(\omega)] + \mathcal{O}(g_B^6 \epsilon, g_B^8), \end{aligned} \quad (3.18)$$

$$\begin{aligned} \frac{-\rho_\chi(\omega)}{16d_A c_\chi^2} &= (1 - 2\epsilon)(1 - \epsilon) \left\{ g_B^4 \rho_{\mathcal{J}_b}(\omega) + 2g_B^6 N_c [3\rho_{\mathcal{I}_h}(\omega) + 2\rho_{\mathcal{I}_i}(\omega)] \right\} \\ &\quad - 2g_B^6 N_c [2\rho_{\mathcal{I}_d}(\omega) + 3\rho_{\mathcal{I}_f}(\omega) + \rho_{\mathcal{I}_j}(\omega) - 2\epsilon \rho_{\mathcal{I}_g}(\omega)] + \mathcal{O}(g_B^6 \epsilon, g_B^8). \end{aligned} \quad (3.19)$$

The structures within the curly brackets on the first rows turn out finite after the insertion of the bare gauge coupling from eq. (2.2) and the re-expansion of the result in terms of the renormalized coupling; therefore the prefactors $1 - \epsilon$, $1 - 2\epsilon$ can actually be set to unity.

3.6. Final expression and its limits

After renormalization and setting $\epsilon \rightarrow 0$ as outlined in sec. 3.5, we obtain our final results for the spectral functions. Furthermore, if they are only needed in the OPE-regime $\omega \gg \pi T$, then all master spectral functions can be determined in a closed form (cf. ref. [36] for general methods as well as appendices A and B for specific results for the structures in eqs. (3.4)–(3.15)). Moreover, the general outcome can be given an interpretation in terms of thermal contributions to “condensates”, cf. ref. [20], which guarantees the cancellation of terms $\propto T^2$, if the lowest-dimensional gauge-invariant condensate has dimensionality four, as is the case in our study. (If scalar fields are present, then there can be a contribution proportional to the condensate $\sim \langle \phi^\dagger \phi \rangle$, implying that terms $\propto T^2$ need not cancel; see e.g. ref. [14]).

The situation is considerably more subtle in the infrared regime, $\omega \ll \pi T$. In fact, at small frequencies, $\omega \sim gT$, the NLO corrections become as large as the leading-order (LO) terms, indicating a breakdown of the perturbative series; this will be discussed with our specific examples in sec. 5.2. In this situation perturbation theory needs to be resummed through effective field theory techniques. For $\omega \sim gT$ the relevant framework is that of the

Hard Thermal Loop effective theory (ref. [38] and references therein), whereas for smaller frequencies still, needed e.g. for transport coefficients, a further re-organization is required (ref. [34] and references therein). The different infrared frequency scales playing a role in the scalar channel have been elaborated upon in ref. [25].

4. Results

Inserting the spectral functions $\rho_{\mathcal{J}_b}$, $\rho_{\mathcal{I}_h}$, $\rho_{\mathcal{I}_i}$ from eqs. (B.6), (B.44), (B.56), respectively, as well as the bare gauge coupling from eq. (2.2), all divergences are seen to cancel in eqs. (3.18), (3.19), as already advertised. Making also use of the finite functions $\rho_{\mathcal{I}_d}$, $\rho_{\mathcal{I}_f}$, $\rho_{\mathcal{I}_j}$ from eqs. (B.19), (B.27), (A.57), respectively, as well as the divergent part of $\rho_{\mathcal{I}_g}$ from eq. (B.33), the final results can be expressed as

$$\frac{\rho_\theta(\omega)}{4d_A c_\theta^2} = \frac{\pi\omega^4}{(4\pi)^2} (1 + 2n_{\frac{\omega}{2}}) \left\{ g^4 + \frac{g^6 N_c}{(4\pi)^2} \left[\frac{22}{3} \ln \frac{\bar{\mu}^2}{\omega^2} + \frac{73}{3} + 8\phi_T(\omega) \right] \right\} + \mathcal{O}(g^8), \quad (4.1)$$

$$\frac{-\rho_\chi(\omega)}{16d_A c_\chi^2} = \frac{\pi\omega^4}{(4\pi)^2} (1 + 2n_{\frac{\omega}{2}}) \left\{ g^4 + \frac{g^6 N_c}{(4\pi)^2} \left[\frac{22}{3} \ln \frac{\bar{\mu}^2}{\omega^2} + \frac{97}{3} + 8\phi_T(\omega) \right] \right\} + \mathcal{O}(g^8), \quad (4.2)$$

where we have introduced

$$n_E \equiv n_B(E), \quad (4.3)$$

with $n_B(E) \equiv 1/(e^{\beta E} - 1)$ denoting the Bose distribution. To this order the only difference between the two channels is in the constant next to the logarithm. The logarithms and the constants next to them agree with those found within the OPE-regime in ref. [36]. (In principle they were computed already in ref. [39], however in a different regularization scheme.) The “non-trivial” T -dependence resides inside the function ϕ_T , which is given by (we have substituted $q = \omega\sigma/2$, $r = \omega\tau/2$ in the expressions of appendix A and B, and defined $\hat{n}_x \equiv n_{\frac{\omega x}{2}}$)

$$\begin{aligned} \phi_T(\omega) = & \int_0^{\frac{1}{2}} d\sigma \hat{n}_\sigma \left\{ \left[\frac{1}{\sigma} - \frac{1}{\sigma-1} - 2 + \sigma - \sigma^2 \right] \ln(1-\sigma) \right. \\ & + \left[\frac{1}{\sigma} - \frac{1}{\sigma+1} + 2 + \sigma + \sigma^2 \right] \ln(1+\sigma) \\ & \left. + \frac{11}{12} \left[\frac{1}{\sigma+1} + \frac{1}{\sigma-1} \right] + \frac{5\sigma}{6} \right\} \\ & + \int_{\frac{1}{2}}^1 d\sigma \hat{n}_\sigma \left\{ \left[\frac{1}{\sigma} - \frac{2}{\sigma-1} - \frac{11}{4} + \sigma - \frac{3\sigma^2}{2} \right] \ln(1-\sigma) \right. \\ & + \left[\frac{1}{\sigma} - \frac{1}{\sigma+1} + 2 + \sigma + \sigma^2 \right] \ln(1+\sigma) \\ & \left. + \left[\frac{1}{\sigma-1} + \frac{3}{4} + \frac{\sigma^2}{2} \right] \ln(\sigma) \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{11}{12} \frac{1}{\sigma+1} - \frac{1}{3} - 2\sigma - \frac{\sigma^2}{3} \Big\} \\
& + \int_1^\infty d\sigma \hat{n}_\sigma \left\{ 2 \left[\frac{1}{\sigma} - \frac{1}{\sigma-1} - 2 + \sigma - \sigma^2 \right] \ln(\sigma-1) \right. \\
& \quad + \left[\frac{1}{\sigma} - \frac{1}{\sigma+1} + 2 + \sigma + \sigma^2 \right] \ln(1+\sigma) \\
& \quad + \left[\frac{1}{\sigma-1} - \frac{1}{\sigma} + 2 - \sigma + \sigma^2 \right] \ln(\sigma) \\
& \quad \left. + \frac{11}{12} \left[\frac{1}{\sigma+1} - \frac{1}{\sigma} \right] + \frac{23}{12} - \frac{13\sigma}{4} - \frac{11\sigma^2}{12} \right\} \\
& + \int_0^1 d\sigma \int_0^{\frac{1}{2}-|\sigma-\frac{1}{2}|} d\tau \frac{\hat{n}_{1-\sigma} \hat{n}_{\sigma+\tau} (1 + \hat{n}_{1-\tau})}{\hat{n}_\tau^2} \times \\
& \quad \times \left[\frac{1}{\sigma\tau} - \frac{5-4\tau+2\tau^2}{4\sigma} - \frac{5-4\sigma+2\sigma^2}{4\tau} + \frac{3}{2} \right] \\
& + \int_1^\infty d\sigma \int_0^{\sigma-1} d\tau \frac{\hat{n}_{\sigma-1} (1 + \hat{n}_{\sigma-\tau}) (\hat{n}_\sigma - \hat{n}_{\tau+1})}{\hat{n}_\tau \hat{n}_{-1}} \times \\
& \quad \times \left[\frac{1}{\sigma\tau} + \frac{5+4\tau+2\tau^2}{4\sigma} - \frac{5-4\sigma+2\sigma^2}{4\tau} - \frac{3}{2} \right] \\
& + \int_0^\infty d\sigma \int_0^\sigma d\tau \frac{(1 + \hat{n}_{\sigma+1}) \hat{n}_{\sigma+\tau} \hat{n}_{\tau+1}}{\hat{n}_\tau^2} \times \\
& \quad \times \left[\frac{1}{\sigma\tau} + \frac{5+4\tau+2\tau^2}{4\sigma} + \frac{5+4\sigma+2\sigma^2}{4\tau} + \frac{3}{2} \right]. \tag{4.4}
\end{aligned}$$

It is conceivable that some of the structures in eq. (4.4) could be written in a simpler form; given that such rewritings do not allow to reduce the dimensionality of the integration, however, we prefer to display the result in the current form, suitable for numerical evaluation. On this point we should mention that eq. (4.4) is defined in the sense of principal value integration; this implies that the 2nd terms from the 2nd and 3rd structures could be combined into

$$\begin{aligned}
& \int_{\frac{1}{2}}^1 d\sigma \hat{n}_\sigma \left\{ -\frac{2}{\sigma-1} \ln(1-\sigma) \right\} + \int_1^\infty d\sigma \hat{n}_\sigma \left\{ -\frac{2}{\sigma-1} \ln(\sigma-1) \right\} \\
& = -2 \left\{ \int_0^{\frac{1}{2}} \frac{d\sigma}{\sigma} (\hat{n}_{1+\sigma} - \hat{n}_{1-\sigma}) \ln(\sigma) + \int_{\frac{3}{2}}^\infty \frac{d\sigma}{\sigma-1} \hat{n}_\sigma \ln(\sigma-1) \right\}, \tag{4.5}
\end{aligned}$$

or, alternatively, that the range $(1, \infty)$ in the 3rd structure could be reflected to $(1, 0)$ through $\sigma \rightarrow 1/\sigma$ and then combined with the 1st and 2nd structures. In any case the integral is rapidly convergent; the result of eq. (4.4) is illustrated in fig. 2.

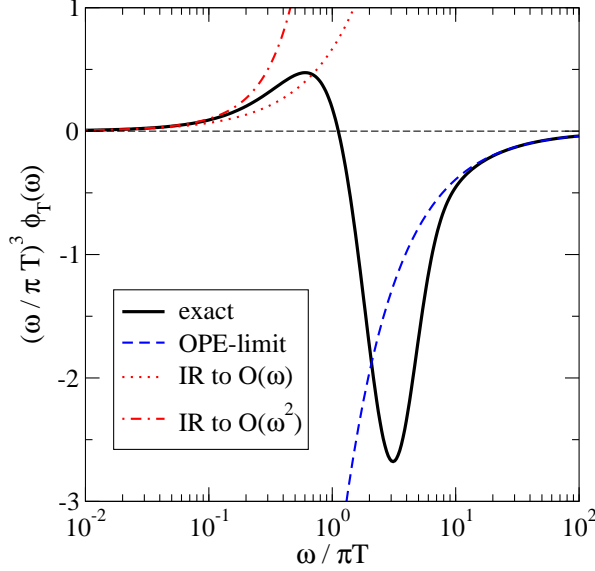


Figure 2: The function ϕ_T from eq. (4.4), multiplied with $(\frac{\omega}{\pi T})^3$, as a function of $\frac{\omega}{\pi T}$. Also shown are the ultraviolet (“OPE”) limit from eq. (5.1) as well as the infrared (“IR”) limit from eq. (5.10).

5. Analysis of the result

5.1. Ultraviolet limit ($\omega \gg \pi T$)

As has already been mentioned the general results simplify considerably in the OPE-limit $\omega \gg \pi T$ and can be expressed in a closed form. Indeed, according to refs. [20, 36], the asymptotic $\omega \gg \pi T$ behaviour of the function ϕ_T from eq. (4.4) should read

$$\phi_T(\omega) \stackrel{\omega \gg \pi T}{\cong} -\frac{176}{45} \frac{\pi^4 T^4}{\omega^4} + \mathcal{O}\left(\frac{\pi^6 T^6}{\omega^6}\right). \quad (5.1)$$

Let us stress again that, as required by OPE [20], all contributions of $\mathcal{O}(\pi^2 T^2/\omega^2)$ from individual graphs need to cancel in the sum, because no gauge-invariant condensate of dimensionality GeV^2 is available. It is easy to verify numerically that the form of eq. (5.1) is indeed approached, cf. fig. 2. The powerlike asymptotic behaviour comes solely from the first integral in eq. (4.4), the other terms falling off exponentially at $\omega \gg \pi T$.

5.2. Infrared limit ($g^2 T/\pi \ll \omega \ll \pi T$)

Another interesting limit is the infrared one, $\omega \ll \pi T$, which shows a much richer structure than the ultraviolet limit. Denoting the six separate parts of eq. (4.4) by I_1, \dots, I_6 , respectively, with the combination appearing in eq. (4.5) taken apart and denoted by I_{23}^{div} , some

work leads to the small- ω expansions

$$I_1 = \left\{ -2 \text{Li}_2\left(\frac{2}{3}\right) - 2 \ln 3 \operatorname{arccoth} 7 - \frac{67 \ln 3}{12} - \ln^2 3 + 6 \ln 2 + \frac{\pi^2}{2} - \frac{1}{6} \right\} \frac{T}{\omega} + \mathcal{O}(\omega) , \quad (5.2)$$

$$I_2 = \left\{ \text{Li}_2\left(\frac{1}{4}\right) + 2 \ln 3 \operatorname{arccoth} 7 + \frac{67 \ln 3}{12} + 3 \ln^2 2 - 8 \ln 2 + \frac{5\pi^2}{8} - 3 \right\} \frac{T}{\omega} + \mathcal{O}(\omega) , \quad (5.3)$$

$$I_3 = -\frac{44\zeta(3)T^3}{3\omega^3} + \left\{ -8 \ln\left(\frac{\omega}{T}\right) - 96 \ln A + 8 \ln(4\pi) + 7 \right\} \frac{\pi^2 T^2}{6\omega^2} \\ + \left\{ -\frac{17}{6} \ln\left(\frac{\omega}{T}\right) - \ln^2 2 + \frac{14 \ln 2}{3} + \frac{11\pi^2}{6} + \frac{37}{12} \right\} \frac{T}{\omega} + \mathcal{O}(\ln \omega) , \quad (5.4)$$

$$I_{23}^{\text{div}} = \left\{ 2 \text{Li}_2\left(\frac{1}{4}\right) - 4 \text{Li}_2\left(\frac{2}{3}\right) - 2 \ln^2 3 + 4 \ln^2 2 - \frac{2\pi^2}{3} \right\} \frac{T}{\omega} + \mathcal{O}(\ln \omega) , \quad (5.5)$$

$$I_4 = \left\{ \frac{\ln 2}{6} - \frac{\pi^2}{8} + \frac{19}{12} \right\} \frac{T}{\omega} + \mathcal{O}(\omega) , \quad (5.6)$$

$$I_5 = \left\{ \ln\left(\frac{\omega}{T}\right) + 12 \ln A - \ln(4\pi) - \frac{7}{6} \right\} \frac{2\pi^2 T^2}{3\omega^2} \\ + \left\{ \ln^2\left(\frac{\omega}{T}\right) - (2 \ln 2 + \frac{1}{3}) \ln\left(\frac{\omega}{T}\right) + c_1 \right\} \frac{T}{\omega} + \mathcal{O}(\ln \omega) , \quad (5.7)$$

$$I_6 = \frac{44\zeta(3)T^3}{3\omega^3} + \left\{ \ln\left(\frac{\omega}{T}\right) + 12 \ln A - \ln(4\pi) + \frac{5}{12} \right\} \frac{2\pi^2 T^2}{3\omega^2} \\ + \left\{ -\ln^2\left(\frac{\omega}{T}\right) + (2 \ln 2 + \frac{19}{6}) \ln\left(\frac{\omega}{T}\right) + c_2 \right\} \frac{T}{\omega} + \mathcal{O}(\ln \omega) , \quad (5.8)$$

where A stands for the Glaisher constant, $\ln A = -\zeta(-1) - \zeta'(-1)$, and we have defined the numerical coefficients

$$c_1 \approx -0.17449 , \quad c_2 \approx -9.32085 . \quad (5.9)$$

Summing together, terms of $\mathcal{O}(T^3/\omega^3)$ as well as all the logarithms cancel, and we are left with the expansion

$$\phi_T(\omega) \stackrel{\omega \ll \pi T}{\approx} \frac{2\pi^2 T^2}{3\omega^2} + 3.31612 \frac{\pi T}{\omega} + \mathcal{O}\left(\ln \frac{\omega}{\pi T}\right) . \quad (5.10)$$

As can be seen from fig. 2, this does agree with a numerical evaluation of ϕ_T for $\omega \ll \pi T$.

Now, even though we can derive a representation of the function ϕ_T for $\omega \ll \pi T$, this does not mean that we know the spectral function in this regime. The reason is that, as can be seen from eqs. (4.1), (4.2) and (5.10), the thermal NLO corrections overtake the LO terms for $\omega \ll gT$. Therefore the naive perturbative expansion breaks down in this regime. The spectral function can be determined only if we find a resummation which re-organizes the perturbative series in a way that a breakdown is avoided.

It is strongly believed that the way to resum the perturbative series around $\omega \sim gT$ goes through the use of the Hard Thermal Loop (HTL) effective theory (ref. [38] and references therein). Although caveats may be difficult to exclude [40], we demonstrate in the following that this framework does indeed reproduce the leading divergence of eq. (5.10) and, through a matching computation, should allow us to postpone the breakdown to $\omega \sim g^2 T/\pi$.

Schematically, matching can be represented as

$$\rho_{\text{resummed}}^{\text{QCD}} = \rho_{\text{resummed}}^{\text{QCD}} - \rho_{\text{resummed}}^{\text{HTL}} + \rho_{\text{resummed}}^{\text{HTL}} \approx \rho_{\text{naive}}^{\text{QCD}} - \rho_{\text{naive}}^{\text{HTL}} + \rho_{\text{resummed}}^{\text{HTL}} . \quad (5.11)$$

Here we have subtracted and added the HTL part, and subsequently noted that if the correct effective theory is used, then the difference between the full and the effective theory computations is infrared safe, so that no resummation is needed for the difference. Since in sec. 4 a “naive” QCD computation was reported, we then need two different HTL computations, one naive and one resummed, in order to obtain the correctly resummed version for QCD.

Some details concerning the HTL computations are given in appendix C. As far as the naive version goes we note that, replacing the Bose distributions through their “classical” limits, i.e. $n_q \rightarrow T/q$, $1 + n_{\omega-q} \rightarrow T/(\omega - q)$, etc., which is equivalent to the physics of HTL resummation [41, 42], the integrals in eq. (C.25) of appendix C.2 are elementary and we get

$$\left. \frac{\rho_{\theta}^{\text{HTL}}(\omega)}{4d_A c_{\theta}^2} \right|_{\text{naive}} = \left. \frac{-\rho_{\chi}^{\text{HTL}}(\omega)}{16d_A c_{\chi}^2} \right|_{\text{naive}} = \frac{\pi g^4 (1 + 2n_{\frac{\omega}{2}})}{(4\pi)^2} \left\{ \omega^4 + \omega^2 m_E^2 \right\} + \mathcal{O}(g^8) . \quad (5.12)$$

Here the Debye mass parameter is defined by

$$m_E^2 \equiv \frac{g^2 N_c T^2}{3} . \quad (5.13)$$

The first term of eq. (5.12) matches the leading-order QCD result in eqs. (4.1), (4.2), whereas the second term exactly corresponds to the leading term of eq. (5.10). So, in the difference $\rho_{\text{naive}}^{\text{QCD}} - \rho_{\text{naive}}^{\text{HTL}}$ appearing in eq. (5.11) the dominant IR divergence drops out.

As far as the resummed version of the HTL computation goes, it is technically similar to the classic dilepton analysis in ref. [43]. The main results are given in eqs. (C.49), (C.50) of appendix C.3. Let us define

$$\left. \frac{\rho_{\theta}^{\text{HTL}}(\omega)}{4d_A c_{\theta}^2} \right|_{\text{resummed}} \equiv \frac{\pi g^4 (1 + 2n_{\frac{\omega}{2}})}{(4\pi)^2} \left\{ \omega^4 + \omega^2 m_E^2 + m_E^4 \phi_{\theta}^{\text{HTL}}(\omega) \right\} , \quad (5.14)$$

$$\left. \frac{-\rho_{\chi}^{\text{HTL}}(\omega)}{16d_A c_{\chi}^2} \right|_{\text{resummed}} \equiv \frac{\pi g^4 (1 + 2n_{\frac{\omega}{2}})}{(4\pi)^2} \left\{ \omega^4 + \omega^2 m_E^2 + m_E^4 \phi_{\chi}^{\text{HTL}}(\omega) \right\} , \quad (5.15)$$

so that the separated terms cancel against the naive version in the difference appearing in eq. (5.11). The result is shown in fig. 3. (The first “cusp” corresponds physically e.g. to a $2 \leftrightarrow 2$ process in which one on-shell “plasmon” gets generated; the second to the 2-plasmon threshold.) For $\omega \lesssim m_E$, $\phi_{\theta, \chi}^{\text{HTL}}$ modifies the IR behaviour of the full expression in a qualitative way (cf. fig. 4 below). For $\omega \gg m_E$, in contrast, $\phi_{\theta, \chi}^{\text{HTL}}$ is a constant of $\mathcal{O}(g^4 m_E^4) \sim \mathcal{O}(g^8 T^4)$ which is vastly overshadowed by the unresummed behaviour $\sim g^4 \omega^4$; therefore in this regime, which is the most important one for us, HTL resummation plays no role.

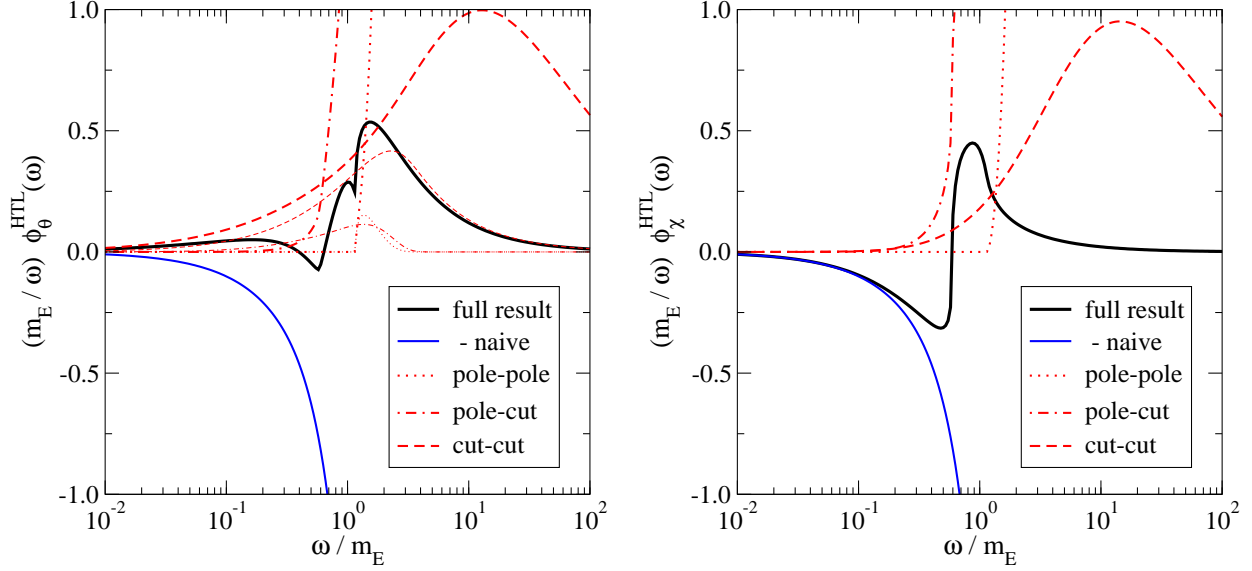


Figure 3: The functions $\phi_{\theta,\chi}^{\text{HTL}}$ from eqs. (5.14), (5.15), multiplied with $\frac{m_E}{\omega}$ (so that the same number of overall ω 's has been factored out as in fig. 2). Also shown are partial contributions from various sub-processes (“- naive” refers to the terms $\omega^4 + \omega^2 m_E^2$ taken apart as in eqs. (5.14), (5.15); in the left panel, thick lines correspond to “transverse” and thin to “electric” modes, respectively).

5.3. Extreme infrared limit ($\omega \lesssim g^2 T/\pi$)

The HTL resummation discussed in sec. 5.2 is supposed to re-organize the perturbative series in a way that it can formally be defined even in the range $\omega \sim gT$. If, however, the frequency is decreased further, down to $\omega \lesssim g^2 T/\pi$, then a further re-organization is needed in order to obtain the correct leading-order result [25]. The “transport peak”, whose height yields the transport coefficient, appears within this regime; in our computation part of the information concerning it is hidden in the so-far omitted structures $\sim \omega^n \delta(\omega)$, to be discussed in sec. 5.6.

5.4. Sum rules

Apart from the UV and IR limits, an interesting crosscheck on the full result is offered by “sum rules”, which concern integrals over the spectral function with a certain weight. These integrals are related to “thermodynamic” Euclidean observables, which can be evaluated independently. Let us elaborate on the information that can be obtained this way.

Because of issues of convergence, some terms need to be subtracted before sum rules can be defined. If we formally subtract the zero-temperature parts from both sides, denoting the results by “ Δ ” (in practice such a subtraction is not without problems, as discussed in

sec. 5.5 below) then, to the order of our computation, the sum rules boil down to [36]

$$-\frac{g^6 N_c T^4}{18} = \frac{\Delta \tilde{G}_\theta(0)}{4d_A c_\theta^2} = 2 \int_0^\infty \frac{d\omega}{\pi\omega} \left[\frac{\Delta \rho_\theta(\omega)}{4d_A c_\theta^2} \right] + \mathcal{O}(g^8), \quad (5.16)$$

$$0 = \frac{\Delta \tilde{G}_\chi(0)}{-16d_A c_\chi^2} = 2 \int_0^\infty \frac{d\omega}{\pi\omega} \left[\frac{-\Delta \rho_\chi(\omega)}{16d_A c_\chi^2} \right] + \mathcal{O}(g^8). \quad (5.17)$$

Here, the thermal parts read (from eqs. (4.1), (4.2))

$$\frac{\Delta \rho_\theta(\omega)}{4d_A c_\theta^2} = \frac{\pi\omega^4}{(4\pi)^2} \left\{ 2n_{\frac{\omega}{2}} \left[g^4 + \frac{g^6 N_c}{(4\pi)^2} \left(\frac{22}{3} \ln \frac{\bar{\mu}^2}{\omega^2} + \frac{73}{3} \right) \right] + (1 + 2n_{\frac{\omega}{2}}) \frac{8g^6 N_c \phi_T(\omega)}{(4\pi)^2} \right\}, \quad (5.18)$$

$$\frac{-\Delta \rho_\chi(\omega)}{16d_A c_\chi^2} = \frac{\pi\omega^4}{(4\pi)^2} \left\{ 2n_{\frac{\omega}{2}} \left[g^4 + \frac{g^6 N_c}{(4\pi)^2} \left(\frac{22}{3} \ln \frac{\bar{\mu}^2}{\omega^2} + \frac{97}{3} \right) \right] + (1 + 2n_{\frac{\omega}{2}}) \frac{8g^6 N_c \phi_T(\omega)}{(4\pi)^2} \right\}. \quad (5.19)$$

Now an immediate “paradox”, raised in ref. [25], is that even though eqs. (5.18), (5.19) have a common positive-definite thermal part at $\mathcal{O}(g^4)$, the integral over this is supposed to vanish, since the left-hand sides of eqs. (5.16), (5.17) are of $\mathcal{O}(g^6)$ or higher. A resolution to this paradox in the *renormalized case*, in which the coupling constant runs, was given in ref. [20]. Beautiful as the argument is, underlining the strength of sum rules in that they allow us to anticipate features of a higher-order computation without carrying it out, the method is cumbersome in our case when we want to use the sum rules as an *exact* crosscheck of an order-by-order computation. In the following we therefore “freeze” the gauge coupling, so that different orders do not mix. The price to pay is that then the issue of “contact terms”, discussed e.g. in ref. [26], needs to be addressed. On the other hand, the contact terms necessarily play a role in the “shear” channel [20, 44], where similar structures appear but without g^4 and running, so the discussion may be useful as an analogue.

Contact terms arise when there is a part in the Euclidean correlators, $\Delta \tilde{G}_\theta(P)$ and $\Delta \tilde{G}_\chi(P)$, which does not vanish as $p_n^2 \rightarrow \infty$. This part is “lost” to the spectral function, and needs to be “added” to the right-hand side of the sum rule, before a comparison with the thermodynamic left-hand side, in which the information is included, can be carried out. (A more precise account of the logic can be found, e.g., in ref. [26].)

From eqs. (4.1) and (4.2) of ref. [36], setting $P = (p_n, \mathbf{0})$ and inserting $\int_{\mathbf{q}} q n_B(q) = \pi^2 T^4/30$, $\int_{\mathbf{q}} n_B(q)/q = T^2/12$, the ultraviolet limits read

$$\frac{\Delta \tilde{G}_\theta(P)}{4d_A c_\theta^2} \stackrel{p_n \gg \pi T}{=} -\frac{4\pi^2 g^4 T^4}{15} \left[1 + \frac{g^2 N_c}{(4\pi)^2} \left(\frac{22}{3} \ln \frac{\bar{\mu}^2}{p_n^2} + \frac{203}{18} \right) \right] + \frac{g^6 N_c T^4}{18} + \mathcal{O}\left(\frac{g^4}{p_n^2}, g^8\right), \quad (5.20)$$

$$\frac{\Delta \tilde{G}_\chi(P)}{-16d_A c_\chi^2} \stackrel{p_n \gg \pi T}{=} -\frac{4\pi^2 g^4 T^4}{15} \left[1 + \frac{g^2 N_c}{(4\pi)^2} \left(\frac{22}{3} \ln \frac{\bar{\mu}^2}{p_n^2} + \frac{347}{18} \right) \right] + \frac{g^6 N_c T^4}{9} + \mathcal{O}\left(\frac{g^4}{p_n^2}, g^8\right), \quad (5.21)$$

where we have kept separate the terms coupling to two different operators in the OPE limit.

At $\mathcal{O}(g^4)$, we now directly observe that if the contact contributions from eqs. (5.20), (5.21) are added to the integrals over the leading-order terms in eqs. (5.18), (5.19), which evaluate to

$$2g^4 \int_0^\infty \frac{d\omega}{\pi\omega} \frac{\pi\omega^4}{(4\pi)^2} 2n_{\frac{\omega}{2}} \stackrel{x=\frac{\omega}{2}}{=} \frac{4g^4}{\pi^2} \int_0^\infty dx x^3 n_x = \frac{4\pi^2 g^4 T^4}{15}, \quad (5.22)$$

then the desired cancellation duly takes place.

As far as $\mathcal{O}(g^6)$ goes, we observe that the constants next to the logarithms in eqs. (5.20), (5.21) can be written as $\frac{203}{18} = \frac{73}{3} - \frac{235}{18}$, $\frac{347}{18} = \frac{97}{3} - \frac{235}{18}$. By making use of eq. (5.22), the parts containing $\frac{73}{3}$ and $\frac{97}{3}$ are seen to exactly cancel against corresponding terms from the integrals over the spectral functions in eqs. (5.18), (5.19). Therefore, the non-trivial sum rule has a common $-\frac{235}{18}$ in both channels. We also observe that the difference in the last terms of eqs. (5.20) and (5.21) precisely explains the difference of the left-hand sides of eqs. (5.16), (5.17). So, we are left with one non-trivial sum rule to check; we choose eq. (5.17).

It requires some care to formulate this sum rule. The reason is the asymptotics of eq. (5.1) which, for a *frozen* coupling, implies that the integral in eq. (5.17) diverges. This is reflected by the appearance of the logarithms involving p_n^2 on the right-hand side of eq. (5.21) (the imaginary part of this logarithm gives the asymptotics of eq. (5.1)). Let us regulate the divergence by restricting ω to $|\omega| \leq \Lambda$. Noting that

$$\text{Im} \left(\ln \frac{1}{p_n^2} \right)_{p_n \rightarrow -i[\omega+i0^+]} = \pi, \quad 2 \int^\Lambda \frac{d\omega}{\pi\omega} \pi = \ln \Lambda^2, \quad (5.23)$$

we deduce that in the contact term we can replace $\ln p_n^2 \rightarrow \ln \Lambda^2$ in the presence of the cutoff. With this logic, the coefficients of $g^6 N_c$ in eqs. (5.17), (5.19) and (5.21) imply the sum rule

$$\begin{aligned} \lim_{\Lambda \rightarrow \infty} \left\{ 2 \int_0^\Lambda \frac{d\omega \omega^3}{(4\pi)^2} \left[2n_{\frac{\omega}{2}} \left(\frac{22}{3} \ln \frac{\bar{\mu}^2}{\omega^2} \right) + 8(1 + 2n_{\frac{\omega}{2}}) \phi_T(\omega) \right] - \frac{4\pi^2 T^4}{15} \left(\frac{22}{3} \ln \frac{\bar{\mu}^2}{\Lambda^2} \right) \right\} \\ = \frac{4\pi^2 T^4}{15} \left(-\frac{355}{18} \right), \end{aligned} \quad (5.24)$$

where $-\frac{355}{18} = -\frac{235}{18} - \frac{15 \times 4}{9}$. We have verified numerically (with a relative error smaller than $\frac{1}{355}$) that this sum rule is satisfied by our function ϕ_T .

Sum rules could also be used in connection with the HTL-results in eqs. (5.14), (5.15). As far as the thermodynamic left-hand sides are concerned, eq. (C.7) yields a contribution $\frac{5}{4\pi} \left(\frac{N_c}{3} \right)^{\frac{3}{2}} g^7 T^4$ to eq. (5.16), which equals the next-to-leading order correction to $T^5 \frac{d}{dT} \left(\frac{e-3p}{T^4} \right)$, whereas eq. (C.8) implies that the left-hand side of eq. (5.17) vanishes even in the presence of HTL-resummation. However, there are again contact terms which need to be added in order to satisfy the sum rules. Given that HTL-resummation is of secondary importance for our study, we refrain from a more detailed discussion here, noting only that a cancellation in the χ -channel is also visually suggested by fig. 3(right).

5.5. Numerical evaluation

After the various crosschecks carried out in secs. 5.1, 5.2 and 5.4, we now move on to the numerical evaluation of our results. The goal is to obtain curves which can in principle be used in the applications outlined in the introduction.

In order to evaluate our result (eqs. (4.1) + (4.2) + (5.11)) numerically, we need to assign a value to the running coupling g^2 . It is only in the regime $\omega \gg \pi T$, where $\phi_T \ll 1$, that two subsequent orders with the same functional form are at our disposal, so that a scale optimization is possible; we then define $\bar{\mu}_{\theta,\chi}^{\text{opt}}$ from the “fastest apparent convergence” criterion based on eqs. (4.1), (4.2):

$$\ln(\bar{\mu}_{\theta}^{\text{opt}(\omega)}) \equiv \ln(\omega) - \frac{73}{44}, \quad \ln(\bar{\mu}_{\chi}^{\text{opt}(\omega)}) \equiv \ln(\omega) - \frac{97}{44}. \quad (5.25)$$

In the infrared regime $\omega \ll \pi T$ we choose g^2 from the “EQCD” setup (cf. ref. [45] and references therein) which at 1-loop order amounts to:

$$\ln(\bar{\mu}_{\theta,\chi}^{\text{opt}(T)}) \equiv \ln(4\pi T) - \gamma_E - \frac{1}{22}. \quad (5.26)$$

For a given ω the larger among these scales is chosen; the switch happens at $\omega \approx 11.3\pi T$ for ρ_{θ} , and at $\omega \approx 19.5\pi T$ for ρ_{χ} . We note that the extremely “late” transition to the vacuum scaling is related to the well-known slow convergence of the perturbative expansion for the Wilson coefficients appearing in the OPE regime. Indeed the fact that in the full computation there is another scale present, πT , which acts as a sort of an infrared cutoff, allows us to obtain a numerical evaluation for frequencies much smaller than in the OPE regime [36]. There is a certain price to pay, however, which is that it is no longer possible for us to split the result into a “zero-temperature” and a “finite-temperature” part, because they have been mixed by the scale choice. (Put another way, the two parts separately are quite sensitive to the confinement scale, but their sum, which we evaluate, is not.)

In order to get an error band for the uncertainty related to the choice of g^2 , we use the 2-loop running of g^2 , with the renormalization scale varied in the range $(0.5 \dots 2.0) \times \bar{\mu}^{\text{opt}}$. The scale parameter, defined as $\Lambda_{\overline{\text{MS}}} \equiv \lim_{\bar{\mu} \rightarrow \infty} \bar{\mu} [b_0 g^2]^{-b_1/2b_0^2} \exp[-\frac{1}{2b_0 g^2}]$, is re-expressed through the critical temperature of the deconfinement transition of pure SU(3) gauge theory; we take this to be $T_c = 1.25\Lambda_{\overline{\text{MS}}}$, which could also be viewed as our definition of “ T_c ”.

The outcome of a numerical evaluation of eqs. (4.1), (4.2), (5.11) is plotted in fig. 4, in units of $\omega^2(\pi T)^2$. We observe that the dependence on the scale choice practically disappears from the NLO result in the ultraviolet domain, $\omega \gg \pi T$. As far as the infrared domain is concerned, we observe an enhancement due to HTL-resummation in the θ -channel akin to that seen in the vector channel [43], whereas in the χ -channel the HTL-resummation removes much of the spectral weight from small frequencies.

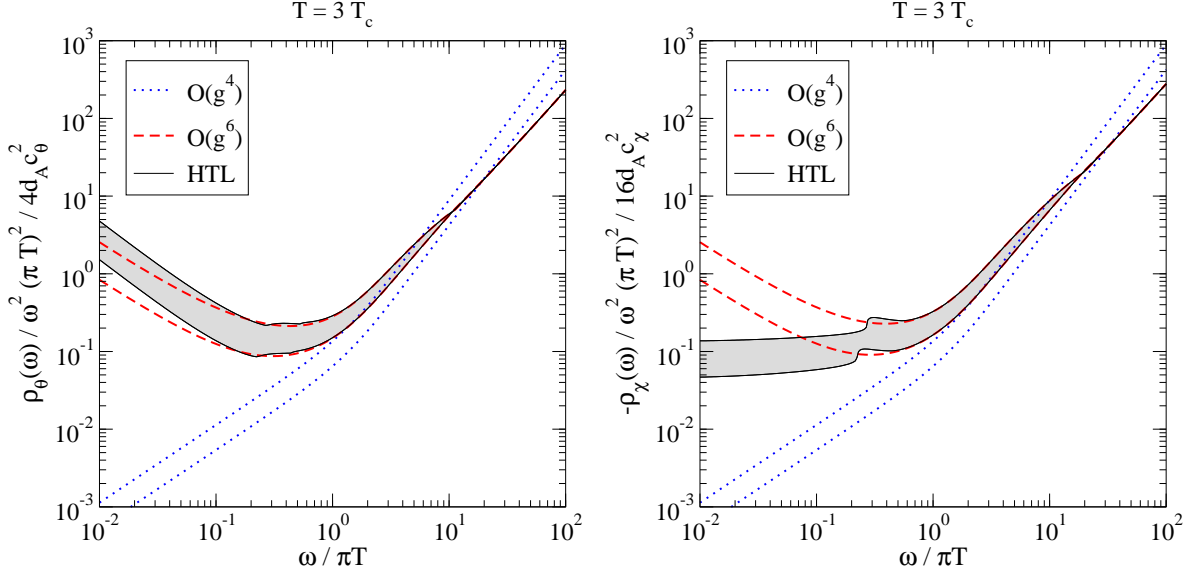


Figure 4: A numerical evaluation of eqs. (4.1), (4.2), (5.11), in units of $\omega^2(\pi T)^2$, for $T = 3.75\Lambda_{\overline{\text{MS}}}$ corresponding to $T = 3T_c$. The gauge coupling has been fixed as explained around eqs. (5.25), (5.26), and the gray band reflects the corresponding uncertainty. (In the $\mathcal{O}(g^4)$ result the “optimal” scale is fixed to the thermal value of eq. (5.26), i.e. does not change with the frequency.)

In ref. [24], a significant “cancellation” in the *thermal part* of the spectral function was advocated for, with a positive domain at small frequencies and a negative domain at large frequencies. For reasons mentioned above, we do not believe our numerical evaluations to be quantitatively reliable for the zero-temperature and finite-temperature parts separately, and therefore abstain from a direct comparison. Nevertheless, we note that on the qualitative level the structure of the finite-temperature part as discussed in ref. [24] (cf. fig. 5 there) agrees perfectly with that of the function ϕ_T as shown in fig. 2.

5.6. Imaginary-time correlators

Although the ultimate goal of a lattice study could be the determination of a transport coefficient, which is by definition a Minkowskian object and obtained from the zero-frequency intercept of $\rho(\omega, \mathbf{0})/\omega$, any practical measurement is carried out in Euclidean signature. It is relatively easy to go from a known Minkowskian spectral function to a Euclidean correlator, but very difficult to take the inverse step; therefore, as a first comparison between lattice data and an analytic computation, it may be helpful to extract the Euclidean correlator from the latter. The principal tool for this is the well-known formula

$$G(\tau) = \int_0^\infty \frac{d\omega}{\pi} \rho(\omega, \mathbf{0}) \frac{\cosh\left(\frac{\beta}{2} - \tau\right) \omega}{\sinh \frac{\beta\omega}{2}}. \quad (5.27)$$

As it turns out, there are some problems hidden in eq. (5.27). One is the issue of “contact terms”, discussed in sec. 5.4; as the name says, one may move them to the Euclidean side of the relation and interpret as a zero-distance term $\propto \delta(\tau)$, giving a finite contribution to $\int_0^\beta d\tau G(\tau)$. Here we stay at non-zero τ , and this issue does not arise. However, there is another issue, namely the possible existence of terms of the type $\sim \omega \delta(\omega)$ in the spectral function which, according to eq. (5.27) (replacing $\int_0^\infty d\omega \rightarrow \frac{1}{2} \int_{-\infty}^\infty d\omega$) can give a constant contribution to the Euclidean correlator. This can be important particularly around the middle of the Euclidean time interval where the absolute value of the correlator is smallest (cf. e.g. ref. [46] and fig. 5 below).

Before proceeding with the specific discussion, let us note that in general the two issues mentioned are somewhat “complementary”. Operators with a higher dimension are more ultraviolet sensitive; this means that, like in many of the sum-integrals in eqs. (3.4)–(3.15), explicit powers of P^2 appear in the numerator and therefore contact contributions, originating from a non-trivial limit of the Euclidean correlators at $p_n^2 \gg (\pi T)^2$, play a role; cf. sec. 5.4. If the operator has a lower dimension, with less or no powers of P^2 , then the ultraviolet regime is safer; on the other hand, the issue with zero-mode contributions is more likely to arise. (Only in exceptional cases are both problems avoided [47, 19].)

Let us recall a simple illustration of the issue. Consider the structure defined in eq. (3.5), but *without* the P^4 in the numerator,

$$\mathcal{J}_b(P) \equiv \not\int_Q \frac{1}{Q^2(Q-P)^2} . \quad (5.28)$$

Carrying out the Matsubara sum and taking the imaginary part leads to (we keep $\mathbf{p} \neq \mathbf{0}$ for the moment; $E_{qp} \equiv |\mathbf{q} - \mathbf{p}|$)

$$\begin{aligned} \rho_{\mathcal{J}_b}(\omega, \mathbf{p}) &= \int_{\mathbf{q}} \frac{\pi}{4qE_{qp}} \left\{ \left[\delta(\omega - q - E_{qp}) - \delta(\omega + q + E_{qp}) \right] (1 + n_q + n_{qp}) \right. \\ &\quad \left. + \left[\delta(\omega - q + E_{qp}) - \delta(\omega + q - E_{qp}) \right] (n_{qp} - n_q) \right\} . \end{aligned} \quad (5.29)$$

(This is like eq. (B.4) but now with the second term added.) For $\omega \rightarrow 0$ the first row gives no contribution because of vanishing phase space, whereas on the second row we can write

$$\begin{aligned} &\left[\delta(\omega - q + E_{qp}) - \delta(\omega + q - E_{qp}) \right] (n_{qp} - n_q) \\ &= \delta(\omega - q + E_{qp}) (n_{q-\omega} - n_q) - \delta(\omega + q - E_{qp}) (n_{q+\omega} - n_q) \\ &\stackrel{\mathbf{p} \rightarrow \mathbf{0}}{\approx} -2\omega \delta(\omega) n'_q = 2\beta\omega \delta(\omega) n_q (1 + n_q) . \end{aligned} \quad (5.30)$$

Inserting this into eq. (5.27) (with $\int_0^\infty d\omega \rightarrow \frac{1}{2} \int_{-\infty}^\infty d\omega$) yields

$$\delta G(\tau) = \int_{-\infty}^\infty \frac{d\omega}{2\pi} \int_{\mathbf{q}} \frac{\pi}{4q^2} 2\beta\omega \delta(\omega) n_q (1 + n_q) \frac{2}{\beta\omega} = \int_{\mathbf{q}} \frac{1}{2q^2} n_q (1 + n_q) , \quad (5.31)$$

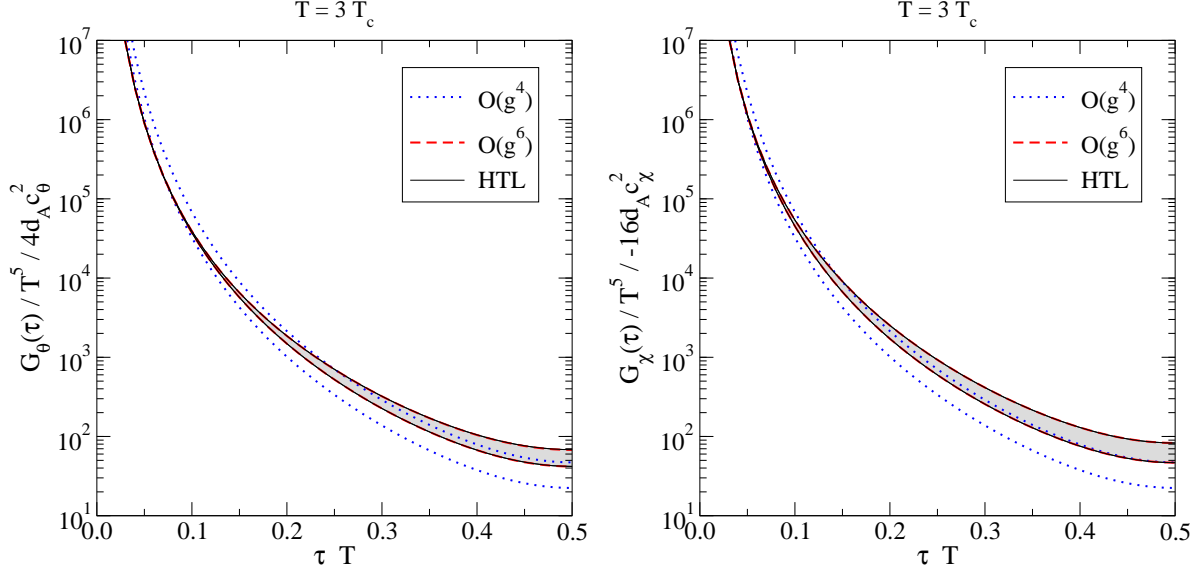


Figure 5: A numerical evaluation of G_θ (left) and G_χ (right), in units of T^5 , for $T = 3.75\Lambda_{\overline{\text{MS}}}$ corresponding to $T = 3T_c$. The gauge coupling has been fixed as explained in fig. 4, and the gray band reflects the corresponding uncertainty.

which would yield a constant (τ -independent) contribution of the type advertised (in this case it is even infrared divergent).

In our situation, however, eq. (5.28) contains an additional P^4 or, after analytic continuation, ω^4 . This kills the contribution from $\omega\delta(\omega)$ at $\mathcal{O}(g^4)$. At higher orders, the loop expansion may introduce additional factors of $g^2 n_\omega \approx g^2 T/\omega$ or, if there is a factorized “hard thermal loop” involved, even $g^2 T^2/\omega^2$. Indeed it is expected that at $\mathcal{O}(g^8)$ a contribution $\sim \omega\delta(\omega)$ arises to ρ_θ , proportional to the heat capacity and related to the fact that $\int_{\mathbf{x}} \theta$ has an overlap with a conserved charge (energy) [25, 26, 24]. At our order, $\mathcal{O}(g^6)$, no such contributions are present and this is the reason that we have omitted from the outset all structures $\sim \omega^n \delta(\omega)$ in appendices A and B.

Even if there are no contributions to $G(\tau)$ from $\omega\delta(\omega)$ -peaks, then how about the infrared structures at small but non-zero frequencies, $\omega \lesssim gT$, discussed in secs. 5.2, 5.3? We note that, parametrically, the range $\omega \lesssim m_E$ gives a contribution to eq. (5.27) of magnitude

$$\delta G(\tau, \mathbf{0}) \sim \int_0^{m_E} \frac{d\omega}{\pi} \frac{T \rho(\omega, \mathbf{0})}{\omega} \sim g^4 m_E^3 T^2 \sim \mathcal{O}(g^7 T^5), \quad (5.32)$$

where we made use of the estimate $\rho(\omega, \mathbf{0}) \sim g^4 m_E^2 T \omega$ from sec. 5.2. This is formally of higher order than the ultraviolet contribution, which is $\mathcal{O}(g^6 T^5)$, but nevertheless more important than a full 3-loop analysis, of $\mathcal{O}(g^8 T^5)$. (The estimate in eq. (5.32) also represents the area delineated by the transport peak and the ω -axis [25].)

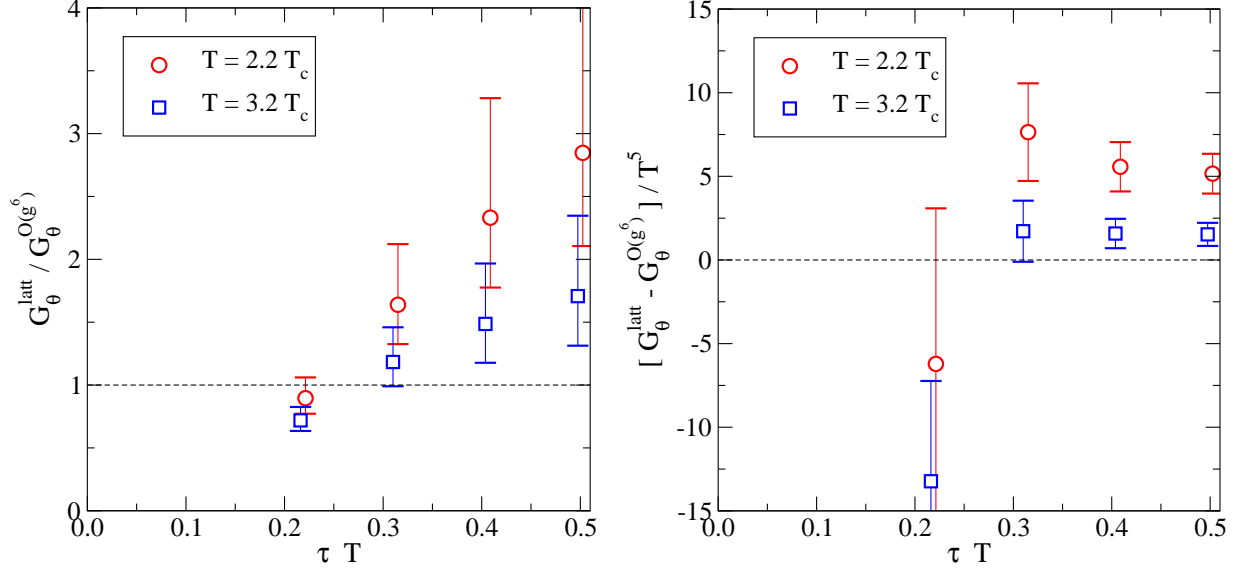


Figure 6: The ratio (left) and difference (right) of the lattice data from fig. 3(left) of ref. [24] and our results from fig. 5(left). The symbols have been slightly displaced for better visibility; the error bars have been obtained by varying both results independently within their uncertainties. Lattice errors are statistical only; discretization artifacts could affect the difference at the shortest distance, where a significant cancellation takes place, the absolute value being about $G_{\theta}/T^5 \sim 50$ there.

After these lengthy qualifications, we apply eq. (5.27) to the results in eqs. (4.1), (4.2), (5.11). In accordance with the discussion around eq. (5.32), the significant difference of the spectral functions at $\mathcal{O}(g^4)$ and $\mathcal{O}(g^6)$ in the infrared regime, visible in fig. 4, converts to a rather mild enhancement of the NLO Euclidean correlator around the center of the Euclidean time interval in fig. 5. Even though ρ_{χ} is suppressed relative to ρ_{θ} at small frequencies, the enhancement is larger in G_{χ} , because of a contribution from large frequencies.

Lattice Monte Carlo results for G_{θ}/T^5 in pure SU(3) gauge theory have been presented in ref. [24]. Multiplying our values by $4d_A c_{\theta}^2 \approx 8(\frac{11}{(4\pi)^2} + \frac{51g^2}{(4\pi)^4})^2 \approx 0.043$, we can carry out a direct comparison; the results are shown in fig. 6. (We use the $\mathcal{O}(g^6)$ rather than the HTL result here, because it gives a simpler partial contribution to the transport coefficient, cf. eq. (5.10); however the difference is barely visible, cf. fig. 5.) As far as the ratio is concerned, we are quite impressed by the good semi-quantitative agreement at short distances. For the difference, the good news is that after the subtraction a function is obtained which no longer shows a visible short-distance divergence; this implies that a model-independent analytic continuation could in principle be attempted. Obviously, however, the significance loss caused by the subtraction implies that it remains a tremendous challenge to obtain the per-mille accuracy level that might turn out to be necessary for this [6]. (There is the further challenge that, as discussed above, a constant τ -independent part in G_{θ} corresponds to a term $\sim \omega \delta(\omega)$ in the spectral

function and does not contribute to the transport coefficient; the resolution has to be good enough to show a statistically significant τ -dependence in the subtracted correlator.)

6. Conclusions and outlook

Even though our study has centered around a specific example, defined in sec. 2 and concerning the “bulk” and “topology” channels of pure Yang-Mills theory, its wider point, as explained in sec. 3, has been to elaborate on a general method for computing next-to-leading order thermal spectral functions for composite operators. We have shown how the method described yields a relatively simple result, eqs. (4.1)–(4.4), which is well suited for a numerical evaluation (fig. 4); as well as how the result can be crosschecked, resummed, and applied in a number of ways (sec. 5). Hopefully the step-by-step description of the labour-intensive phase of the procedure (appendix A for the most complicated “master” sum-integral appearing at 2-loop level), will also turn out to lend itself to a number of generalizations; the simplest of them may be the inclusion of a non-trivial mass spectrum, which in a specific case has in fact already been achieved for many intermediate stages.

In the course of the computation we have shown that various divergences, sometimes classified as being of “soft”, “collinear”, or “thermal” origin, cancel in every of our “master” spectral functions separately. That this happens for the sum is in accordance with the general OPE analysis of ref. [20], showing the absence of infrared divergences up to 4-loop level in the regime $\omega \gg \pi T$, and also with a NLO analysis of the vector current correlator at $\omega \sim \pi T$, where the cancellation of divergences was verified long ago [15]–[17]. It also agrees with the specific discussion of another correlator in ref. [14] for the limit $\omega \gg \pi T$, which found terms proportional to T^2 to be finite at NLO. However, our results are valid down to $\omega \sim \pi T$ and somewhat below it, and for any single 2-loop master spectral function separately.

For small frequencies, $\omega \ll \pi T$, the naive perturbative series does break down. Nevertheless, as long as $g^2 T/\pi \ll \omega \ll \pi T$, it can (probably) still be “repaired” through a Hard Thermal Loop resummation (sec. 5.2). The price for the resummation is that in this regime only one order of the weak-coupling series is available, and NLO corrections could well be substantial [48]. Once $\omega \lesssim g^2 T/\pi$, even the resummed result loses its validity, and other techniques are needed (cf. sec. 5.3); this regime is not addressed in the present study.

Even if the small-frequency regime of a spectral function suffers from infrared problems, the corresponding imaginary-time correlator can still be computed and compared directly with lattice simulations, because the contribution emerges dominantly from ultraviolet frequencies, $\omega \gtrsim \pi T$ (cf. sec. 5.6). The ultimate use envisaged for such results within thermal QCD is that the perturbative imaginary-time correlator can be subtracted from a lattice measurement; the remainder could be ultraviolet finite and only sensitive to thermal infrared physics, hence showing relatively little “structure”; and that therefore the spectral function corresponding

to the remainder could be reconstructed with less model input than without the subtraction. That such a program may be feasible is demonstrated by fig. 6(right) for the bulk channel, although much further work is needed before the quality of the remainder is satisfactory.

As promised in the introduction we finally comment on cosmological applications. In the present paper the spectral functions were computed at vanishing spatial momentum, i.e. for $\mathcal{P} = (\omega, \mathbf{0})$ in the notation of the introduction, or $P = (-i\omega, \mathbf{0})$ in the Euclidean notation used elsewhere. If, in contrast, the pseudoscalar correlator were computed “on-shell”, i.e. for $P = (-iE_p, \mathbf{p})$, then it would be directly proportional to the production rate of axions (see, e.g., ref. [49]), which are among standard Dark Matter candidates. In the case of a scalar density, one could similarly envisage a production rate of dilatons, although in this case it might be natural to assume the dilatons to further decay, perhaps into particle–antiparticle pairs in a “hidden sector”. Then the dilaton itself could be off-shell, and our result would yield the production rate of the hidden-sector pair, in the same way as the spectral function of the electromagnetic current in QCD with $P = (-i\omega, \mathbf{0})$ yields the production rate of a dilepton pair with a total energy equal to ω (cf. ref. [15]). As far as the corresponding transport coefficients are concerned, that related to the pseudoscalar density yields for $N_c = 2$ the baryon number violation rate which is certainly important for cosmology [50] (even the case $N_c = 3$ may have some relevance, cf. the discussion in ref. [35] and references therein), whereas bulk viscosity could determine e.g. a moduli decay rate [51, 52]. Beyond this, the sum-integrals that we have considered are similar to those appearing in other production rates, e.g. that of right-handed fermions [53], and so (with the transformation of some lines to fermions, which goes simply by changing $n_B \rightarrow -n_F$), we envisage that our techniques extended to $P = (-iE_p, \mathbf{p})$ might find further applications. In fact, in ref. [14], the results were approximated by setting $P = (-iM, \mathbf{0})$, and then the kinematics is identical to ours. We hope to be able to apply our methods to some of these problems in future work.

Acknowledgements

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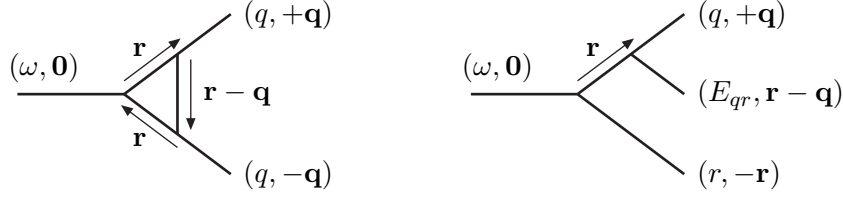


Figure 7: Examples of physical amplitudes contained in eq. (A.4). Left: “virtual corrections”, or “factorized integrals”. Right: “real corrections”, or “phase space integrals”. The whole spectral function amounts essentially to the absolute value squared of an amplitude, and the virtual process contributes through its interference with a tree-level process. (In the case shown, the final answer vanishes in the zero-temperature limit, but remains finite for $T \neq 0$.)

A. Detailed procedure with the example of $\rho_{\mathcal{I}_j}$

Let us consider the master sum-integral defined as (this is a slight generalization of eq. (3.15))

$$\mathcal{I}_j(P) \equiv \oint_{Q,R} \frac{P^6}{Q^2 R^2 [(Q-R)^2 + \lambda^2] (Q-P)^2 (R-P)^2} . \quad (\text{A.1})$$

The corresponding spectral function, evaluated at vanishing spatial momentum, is defined by

$$\rho_{\mathcal{I}_j}(\omega) \equiv \text{Im} \left[\mathcal{I}_j(P) \right]_{P \rightarrow (-i[\omega+i0^+], \mathbf{0}} . \quad (\text{A.2})$$

In the following we give some details on how $\rho_{\mathcal{I}_j}$ can be given a rapidly convergent 2-dimensional integral representation. Since $\rho_{\mathcal{I}_j}$ is the most complicated structure encountered, the same techniques will also apply to the other cases. (The only major simplification in $\rho_{\mathcal{I}_j}$ is that it is ultraviolet finite so that the analysis can be carried out without ultraviolet regularization; nevertheless, the part which contains ultraviolet divergences in other cases will already be clearly identified, cf. sec. A.2.)

A.1. Matsubara sums

As a first step we carry out the Matsubara sums in eq. (A.1);¹ take the discontinuity leading to the spectral function; and set $\mathbf{p} \rightarrow \mathbf{0}$. The procedure is similar to that described in some detail in refs. [18, 19]. Like in ref. [19] it is useful to “regulate” some of the propagators by introducing a small mass parameter which is set to zero in the end (λ^2 in eq. (A.1)). To keep the situation maximally symmetric, it is convenient to choose the propagator carrying the momentum $Q - R$ for this task. Denoting

$$E_q \equiv q , \quad E_r \equiv r , \quad E_{qr} \equiv \sqrt{(\mathbf{q} - \mathbf{r})^2 + \lambda^2} , \quad (\text{A.3})$$

¹An alternative strategy might consist of performing first the spatial integrals, which can be done for general masses [54], and by then re-interpreting (some of) the masses as Matsubara frequencies, over which a summation is carried out.

and not displaying any terms proportional to $\omega^n \delta(\omega)$ (cf. sec. 5.6), the result from eq. (B.33) of ref. [18] can be re-interpreted as²

$$\begin{aligned}
\rho_{\mathcal{I}_j}(\omega) = & \int_{\mathbf{q}, \mathbf{r}} \frac{\omega^6 \pi}{4qr E_{qr}} \left\{ \right. \\
& \frac{1}{8q^2} \left[\delta(\omega - 2q) - \delta(\omega + 2q) \right] \times \\
& \times \left[\left(\frac{1}{(q+r-E_{qr})(q+r)} - \frac{1}{(q-r+E_{qr})(q-r)} \right) (1+2n_q)(n_{qr}-n_r) \right. \\
& \left. + \left(\frac{1}{(q+r+E_{qr})(q+r)} - \frac{1}{(q-r-E_{qr})(q-r)} \right) (1+2n_q)(1+n_{qr}+n_r) \right] \\
& + \frac{1}{8r^2} \left[\delta(\omega - 2r) - \delta(\omega + 2r) \right] \times \\
& \times \left[\left(\frac{1}{(q+r-E_{qr})(q+r)} - \frac{1}{(q-r-E_{qr})(q-r)} \right) (1+2n_r)(n_{qr}-n_q) \right. \\
& \left. + \left(\frac{1}{(q+r+E_{qr})(q+r)} - \frac{1}{(q-r+E_{qr})(q-r)} \right) (1+2n_r)(1+n_{qr}+n_q) \right] \\
& + \left[\delta(\omega - q - r - E_{qr}) - \delta(\omega + q + r + E_{qr}) \right] \frac{(1+n_{qr})(1+n_q+n_r) + n_q n_r}{(q+r+E_{qr})^2 (q-r+E_{qr})(q-r-E_{qr})} \\
& + \left[\delta(\omega - q - r + E_{qr}) - \delta(\omega + q + r - E_{qr}) \right] \frac{n_{qr}(1+n_q+n_r) - n_q n_r}{(q+r-E_{qr})^2 (q-r+E_{qr})(q-r-E_{qr})} \\
& + \left[\delta(\omega - q + r - E_{qr}) - \delta(\omega + q - r + E_{qr}) \right] \frac{n_r(1+n_q+n_{qr}) - n_q n_{qr}}{(q-r+E_{qr})^2 (q+r+E_{qr})(q+r-E_{qr})} \\
& \left. + \left[\delta(\omega + q - r - E_{qr}) - \delta(\omega - q + r + E_{qr}) \right] \frac{n_q(1+n_r+n_{qr}) - n_r n_{qr}}{(q-r-E_{qr})^2 (q+r+E_{qr})(q+r-E_{qr})} \right\}.
\end{aligned} \tag{A.4}$$

The first two structures, with simple δ -function constraints, will technically be referred to as “factorized” integrals but correspond physically to virtual corrections; the latter four structures, with more complicated δ -constraints, are technically labelled “phase space” integrals but correspond physically to real processes (some examples are illustrated in fig. 7). Among the factorized integrals a further subdivision into two classes is possible, namely to “powerlike” integrals in which the (unconstrained) integration proceeds like at zero temperature, without any Boltzmann suppression at large momenta (these are virtual vacuum corrections); as well as to “exponential” integrals, which contain Bose distributions and are guaranteed to be ultraviolet convergent (these are virtual thermal corrections). We first discuss these three classes separately, and then collect together the results.

It is important to note that the factorized and phase space integrals are both divergent (or ill-defined) for $\lambda \rightarrow 0$, because of the appearance of poles in some of the denominators.

²We have also rederived the expression from scratch, using the same techniques as in ref. [18] but applied directly to the bosonic case, with all propagators taking their tree-level forms.

For example in ref. [14] various divergences were classified as being of “soft”, “collinear”, or “thermal infrared” origin (the last coming from n_E at small E). We do not separately keep track of the nature of the divergences, but their eventual cancellation, when the real and virtual corrections are added together, works out like in ref. [14], and is the reason that we can take $\lambda \rightarrow 0$ in the end; however we establish the cancellation for any value of ω/T , whereas in ref. [14] only terms of relative magnitude T^2/ω^2 in the OPE-limit were analyzed. (Of course, similar cancellations were seen long ago in refs. [15]–[17].) In practice we add as a further regulator a principle value prescription for the poles; then changes of integration variables, such as $q \leftrightarrow r$, are unproblematic and allow to simplify the expressions somewhat.

A.2. Virtual vacuum corrections

Starting with the factorized part, it is immediately clear that the symmetry $q \leftrightarrow r$ allows us to combine the two first terms of eq. (A.4). Let us pick the latter case, with $\delta(\omega - 2r)$ for $\omega > 0$, as a representative. The integral over \mathbf{r} is trivial,

$$\int_{\mathbf{r}} \pi \delta(\omega - 2r) = \frac{\omega^2}{16\pi}, \quad (\text{A.5})$$

and $1/(8r^3)$ from the original “measure” goes over into $1/\omega^3$. In the remaining \mathbf{q} -integral we substitute the angular variable through E_{qr} :

$$\int_{\mathbf{q}} \frac{1}{2qE_{qr}} = \frac{1}{4\pi^2\omega} \int_0^\infty dq \int_{E_{qr}^-}^{E_{qr}^+} dE_{qr}, \quad E_{qr}^\pm \equiv \sqrt{\left(q \pm \frac{\omega}{2}\right)^2 + \lambda^2}. \quad (\text{A.6})$$

Thereby the factorized powerlike integral can be expressed as

$$\rho_{\mathcal{I}_j}^{(\text{fz,p})}(\omega) \equiv \frac{\omega^4}{(4\pi)^3} (1 + 2n_{\frac{\omega}{2}}) \int_0^\infty dq \int_{E_{qr}^-}^{E_{qr}^+} dE_{qr} \left\{ \mathbb{P} \left[\frac{1}{(q + \frac{\omega}{2} + E_{qr})(q + \frac{\omega}{2})} - \frac{1}{(q - \frac{\omega}{2} + E_{qr})(q - \frac{\omega}{2})} \right] \right\}. \quad (\text{A.7})$$

The integration range is illustrated in fig. 8.

Now, we could immediately integrate over the variable E_{qr} in eq. (A.7). It turns out to be somewhat simpler to take the limit $\lambda \rightarrow 0$, however, if we change the order of integrations (the same change will also be useful in sec. A.3). This can be achieved through

$$\int_0^\infty dq \int_{\sqrt{(q - \frac{\omega}{2})^2 + \lambda^2}}^{\sqrt{(q + \frac{\omega}{2})^2 + \lambda^2}} dE_{qr} = \int_\lambda^\infty dE_{qr} \int_{|\frac{\omega}{2} - \sqrt{E_{qr}^2 - \lambda^2}|}^{\frac{\omega}{2} + \sqrt{E_{qr}^2 - \lambda^2}} dq. \quad (\text{A.8})$$

Subsequently we rename the inner integration variable to be r , the outer one to be q . Partial

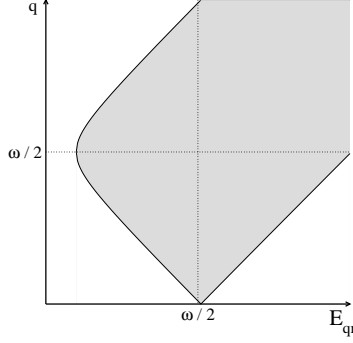


Figure 8: The integration range for the factorized integrals, cf. eq. (A.6), for $\lambda = \omega/10$.

fractioning with respect to the new r , eq. (A.7) becomes

$$\rho_{\mathcal{I}_j}^{(\text{fz,p})}(\omega) = \frac{\omega^4}{(4\pi)^3} (1 + 2n_{\frac{\omega}{2}}) \int_{\lambda}^{\infty} \frac{dq}{q} \int_{|\frac{\omega}{2} - \sqrt{q^2 - \lambda^2}|}^{\frac{\omega}{2} + \sqrt{q^2 - \lambda^2}} dr \left\{ \mathbb{P} \left[\frac{1}{r + \frac{\omega}{2}} - \frac{1}{q + r + \frac{\omega}{2}} - \frac{1}{r - \frac{\omega}{2}} + \frac{1}{q + r - \frac{\omega}{2}} \right] \right\}. \quad (\text{A.9})$$

Carrying out the integrals yields

$$\rho_{\mathcal{I}_j}^{(\text{fz,p})}(\omega) = \frac{\omega^4}{(4\pi)^3} (1 + 2n_{\frac{\omega}{2}}) \left\{ \int_{\lambda}^{\sqrt{(\frac{\omega}{2})^2 + \lambda^2}} \frac{dq}{q} \ln \left| \frac{q + \sqrt{q^2 - \lambda^2}}{q - \sqrt{q^2 - \lambda^2}} \right| \left| \frac{\omega + \sqrt{q^2 - \lambda^2}}{\omega - \sqrt{q^2 - \lambda^2}} \right| \left| \frac{\omega + q - \sqrt{q^2 - \lambda^2}}{\omega + q + \sqrt{q^2 - \lambda^2}} \right| \right. \\ \left. + \int_{\sqrt{(\frac{\omega}{2})^2 + \lambda^2}}^{\infty} \frac{dq}{q} \ln \left| \frac{(q + \sqrt{q^2 - \lambda^2})^2}{q^2 - \lambda^2} \right| \left| \frac{\sqrt{q^2 - \lambda^2} + \omega}{q + \sqrt{q^2 - \lambda^2} + \omega} \right| \left| \frac{\sqrt{q^2 - \lambda^2} - \omega}{q + \sqrt{q^2 - \lambda^2} - \omega} \right| \right\}, \quad (\text{A.10})$$

where $\sqrt{(\frac{\omega}{2})^2 + \lambda^2}$ is the value of q for which the lower limit $|\frac{\omega}{2} - \sqrt{q^2 - \lambda^2}|$ of the r -integration in eq. (A.9) vanishes.

Whereas eq. (A.10) is still exact, we would in the end like to take the limit $\lambda \rightarrow 0$. In most of the terms of eq. (A.10) this can be achieved through a Taylor expansion in λ/q , but if an integration is sensitive to the range $q \sim \lambda$ then this is not possible, and the integrand needs to be kept in its exact form. Analyzing the situation carefully, this can be seen to only happen with the very first term, in the vicinity of the lower bound. Thereby we obtain

$$\rho_{\mathcal{I}_j}^{(\text{fz,p})}(\omega) \approx \frac{\omega^4}{(4\pi)^3} (1 + 2n_{\frac{\omega}{2}}) \left\{ \int_{\lambda}^{\frac{\omega}{2}} \frac{dq}{q} \ln \left| \frac{q + \sqrt{q^2 - \lambda^2}}{q - \sqrt{q^2 - \lambda^2}} \right| + \int_0^{\frac{\omega}{2}} \frac{dq}{q} \ln \left| \frac{(\omega + q)\omega}{(\omega - q)(\omega + 2q)} \right| + \int_{\frac{\omega}{2}}^{\infty} \frac{dq}{q} \ln \left| \frac{4(q^2 - \omega^2)}{4q^2 - \omega^2} \right| \right\}, \quad (\text{A.11})$$

where “ \approx ” indicates that terms of $\mathcal{O}(\lambda \ln \lambda)$ and smaller have been omitted. Remarkably the 2nd and 3rd integrals in eq. (A.11) cancel against each other, so that in total

$$\rho_{\mathcal{I}_j}^{(\text{fz,p})}(\omega) \approx \frac{\omega^4}{(4\pi)^3} (1 + 2n_{\frac{\omega}{2}}) \int_{\lambda}^{\frac{\omega}{2}} \frac{dq}{q} \ln \left| \frac{q + \sqrt{q^2 - \lambda^2}}{q - \sqrt{q^2 - \lambda^2}} \right|. \quad (\text{A.12})$$

A.3. Virtual thermal corrections

For the factorized exponential integrals we proceed in a similar fashion. The change of ordering in eq. (A.8) and the subsequent renaming of variables is only carried out in terms containing the distribution n_{qr} . Thereby the relevant part of eq. (A.4) becomes

$$\begin{aligned} \rho_{\mathcal{I}_j}^{(\text{fz,e})}(\omega) \equiv & \frac{\omega^4}{(4\pi)^3} (1 + 2n_{\frac{\omega}{2}}) \int_0^\infty dq n_q \int_{\sqrt{(q-\frac{\omega}{2})^2 + \lambda^2}}^{\sqrt{(q+\frac{\omega}{2})^2 + \lambda^2}} dr \left\{ \right. \\ & \mathbb{P} \left[\frac{1}{(q+r+\frac{\omega}{2})(q+\frac{\omega}{2})} - \frac{1}{(q+r-\frac{\omega}{2})(q-\frac{\omega}{2})} \right. \\ & \quad \left. \left. + \frac{1}{(q-r-\frac{\omega}{2})(q-\frac{\omega}{2})} - \frac{1}{(q-r+\frac{\omega}{2})(q+\frac{\omega}{2})} \right] \right\} \\ & + \frac{\omega^4}{(4\pi)^3} (1 + 2n_{\frac{\omega}{2}}) \int_{\lambda}^{\frac{\omega}{2}} dq n_q \int_{|\frac{\omega}{2} - \sqrt{q^2 - \lambda^2}|}^{\frac{\omega}{2} + \sqrt{q^2 - \lambda^2}} dr \left\{ \right. \\ & \mathbb{P} \left[\frac{1}{(q+r+\frac{\omega}{2})(r+\frac{\omega}{2})} - \frac{1}{(q+r-\frac{\omega}{2})(r-\frac{\omega}{2})} \right. \\ & \quad \left. \left. + \frac{1}{(q-r+\frac{\omega}{2})(r-\frac{\omega}{2})} - \frac{1}{(q-r-\frac{\omega}{2})(r+\frac{\omega}{2})} \right] \right\}. \quad (\text{A.13}) \end{aligned}$$

The integration over r is again trivial, partial fractioning the latter term like in eq. (A.9). In a few steps we obtain

$$\begin{aligned} \rho_{\mathcal{I}_j}^{(\text{fz,e})}(\omega) = & \frac{\omega^4}{(4\pi)^3} (1 + 2n_{\frac{\omega}{2}}) \left\{ \right. \\ & \int_0^\infty dq n_q \mathbb{P} \left[\frac{1}{q + \frac{\omega}{2}} \ln \left| \frac{\lambda^2}{2q\omega - \lambda^2} \right| + \frac{1}{q - \frac{\omega}{2}} \ln \left| \frac{\lambda^2}{2q\omega + \lambda^2} \right| \right] \\ & + \int_{\lambda}^{\frac{\omega}{2}} dq n_q \left[\frac{1}{q} \ln \left| \frac{q + \frac{\lambda^2}{\omega} + \sqrt{q^2 - \lambda^2}}{q + \frac{\lambda^2}{\omega} - \sqrt{q^2 - \lambda^2}} \right| + \frac{1}{q} \ln \left| \frac{q - \frac{\lambda^2}{\omega} + \sqrt{q^2 - \lambda^2}}{q - \frac{\lambda^2}{\omega} - \sqrt{q^2 - \lambda^2}} \right| \right] \right\}. \quad (\text{A.14}) \end{aligned}$$

A.4. Real corrections

As far as the latter four structures of eq. (A.4) are concerned, we again rewrite the integration measure by substituting the angle between \mathbf{q} and \mathbf{r} through E_{qr} :

$$\int_{\mathbf{q}, \mathbf{r}} \frac{\pi}{4qr E_{qr}} = \frac{2}{(4\pi)^3} \int_0^\infty dq \int_0^\infty dr \int_{E_{qr}^-}^{E_{qr}^+} dE_{qr}, \quad E_{qr}^\pm \equiv \sqrt{(q \pm r)^2 + \lambda^2}. \quad (\text{A.15})$$

It is also convenient to factor out a common $n_q n_r n_{qr}$ from the Bose distributions, yielding

$$(1 + n_{qr})(1 + n_q + n_r) + n_q n_r = n_q n_r n_{qr} (e^{q+r+E_{qr}} - 1) , \quad (\text{A.16})$$

$$n_{qr}(1 + n_q + n_r) - n_q n_r = n_q n_r n_{qr} (e^{q+r} - e^{E_{qr}}) , \quad (\text{A.17})$$

$$n_r(1 + n_q + n_{qr}) - n_q n_{qr} = n_q n_r n_{qr} (e^{q+E_{qr}} - e^r) , \quad (\text{A.18})$$

$$n_q(1 + n_r + n_{qr}) - n_r n_{qr} = n_q n_r n_{qr} (e^{r+E_{qr}} - e^q) , \quad (\text{A.19})$$

and to make use of the δ -functions to simplify the denominators. Choosing furthermore dimensionless units in the following whereby all variables are expressed in terms of T , and noting that, for $0 < \lambda < \omega$, only four of the δ -constraints get realized, we can rewrite the phase space part of the integral as

$$\begin{aligned} \rho_{\mathcal{I}_j}^{(\text{ps})}(\omega) &\equiv \frac{2\omega^4}{(4\pi)^3} \int_0^\infty dq \int_0^\infty dr \int_{E_{qr}^-}^{E_{qr}^+} dE_{qr} n_q n_r n_{qr} \left\{ \right. \\ &\quad (\text{i}) \quad \frac{\delta(\omega - q - r - E_{qr})}{(2r - \omega)(2q - \omega)} \left(1 - e^{q+r+E_{qr}} \right) \\ &\quad (\text{ii}) \quad + \frac{\delta(\omega - q - r + E_{qr})}{(2r - \omega)(2q - \omega)} \left(e^{E_{qr}} - e^{q+r} \right) \\ &\quad (\text{iii}) \quad + \frac{\delta(\omega + q - r - E_{qr})}{(2r - \omega)(2q + \omega)} \left(e^{r+E_{qr}} - e^q \right) \\ &\quad (\text{iv}) \quad + \frac{\delta(\omega - q + r - E_{qr})}{(2r + \omega)(2q - \omega)} \left(e^{q+E_{qr}} - e^r \right) \left. \right\} . \end{aligned} \quad (\text{A.20})$$

Of the integrals here, the ones labelled (iii) and (iv) are related by the symmetry $q \leftrightarrow r$, so essentially only three cases remain. We choose to regard the integral over r as an “inner” integration, to be carried out first. Then, with some work, the following relations can be established for $\omega > \lambda$:

$$\begin{aligned} (\text{i}) \quad &\int_0^\infty dq \int_0^\infty dr \int_{E_{qr}^-}^{E_{qr}^+} dE_{qr} \delta(\omega - q - r - E_{qr}) \phi(q, r, E_{qr}) \\ &= \int_0^{\frac{\omega^2 - \lambda^2}{2\omega}} dq \int_{\frac{\omega(\omega - 2q) - \lambda^2}{2\omega}}^{\frac{\omega(\omega - 2q) - \lambda^2}{2(\omega - 2q)}} dr \phi(q, r, \omega - q - r) , \end{aligned} \quad (\text{A.21})$$

$$\begin{aligned} (\text{ii}) \quad &\int_0^\infty dq \int_0^\infty dr \int_{E_{qr}^-}^{E_{qr}^+} dE_{qr} \delta(\omega - q - r + E_{qr}) \phi(q, r, E_{qr}) \\ &= \int_{\frac{\omega}{2}}^\infty dq \int_{\frac{\omega(2q - \omega) + \lambda^2}{2(2q - \omega)}}^\infty dr \phi(q, r, -\omega + q + r) , \end{aligned} \quad (\text{A.22})$$

$$\begin{aligned} (\text{iii}) \quad &\int_0^\infty dq \int_0^\infty dr \int_{E_{qr}^-}^{E_{qr}^+} dE_{qr} \delta(\omega + q - r - E_{qr}) \phi(q, r, E_{qr}) \\ &= \int_0^\infty dq \int_{\frac{\omega(\omega + 2q) - \lambda^2}{2(\omega + 2q)}}^{\frac{\omega(\omega + 2q) - \lambda^2}{2\omega}} dr \phi(q, r, \omega + q - r) , \end{aligned} \quad (\text{A.23})$$

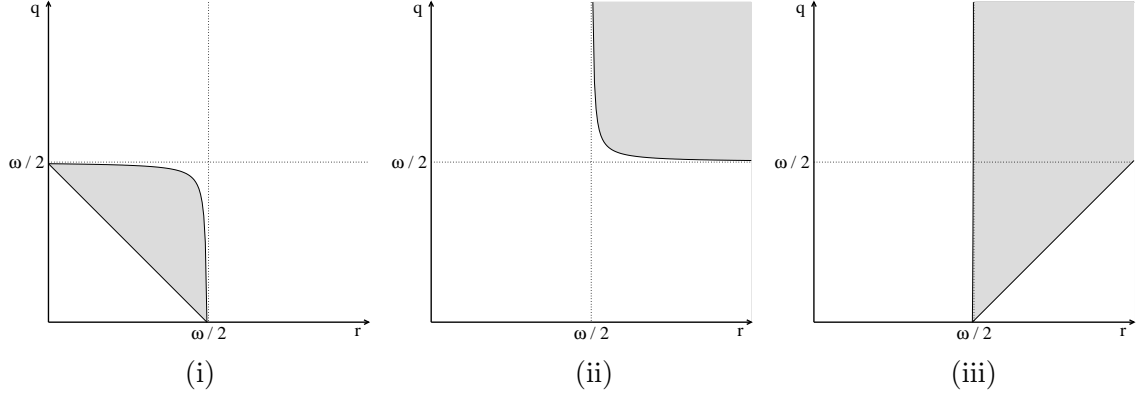


Figure 9: Original integration ranges (eqs. (A.21)–(A.23)) for the phase space integrals, for $\lambda = \omega/10$.

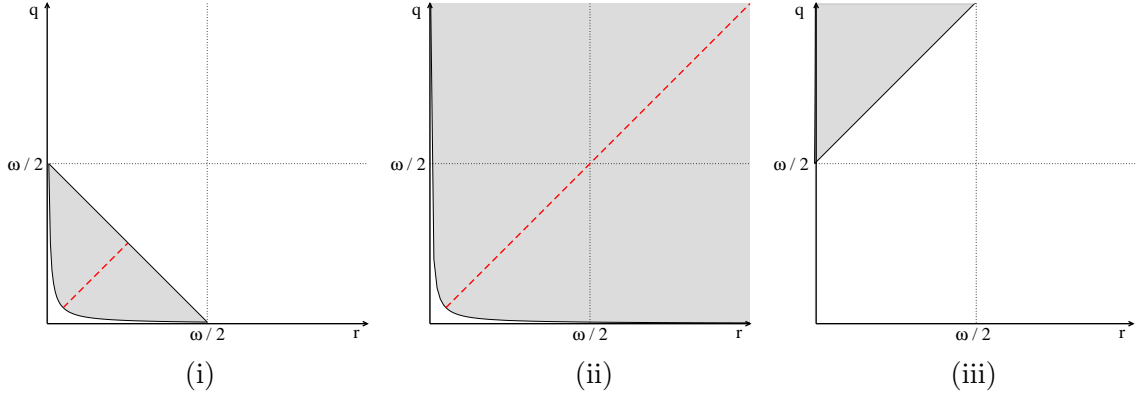


Figure 10: Integration ranges after the shifts in eqs. (A.27)–(A.29), for $\lambda = \omega/10$. The dashed line indicates the diagonal $q = r$, which can be used to reflect all integrations to the octant $q \geq r$.

where ϕ is an arbitrary function. The ranges are illustrated in fig. 9.

It appears advantageous, however, to shift the ranges somewhat, in order to place poles so that their eventual cancellation is easier to see. To this end we undertake shifts which turn all denominators of eq. (A.20) into a universal $1/(4qr)$. The shifts are

$$(i) : \quad q \rightarrow \frac{\omega}{2} - q, \quad r \rightarrow \frac{\omega}{2} - r, \quad (A.24)$$

$$(ii) : \quad q \rightarrow \frac{\omega}{2} + q, \quad r \rightarrow \frac{\omega}{2} + r, \quad (A.25)$$

$$(iii) : \quad q \rightarrow -\frac{\omega}{2} + q, \quad r \rightarrow \frac{\omega}{2} + r, \quad (A.26)$$

and the integration ranges of eqs. (A.21)–(A.23) convert into

$$(i) \quad \int_0^{\frac{\omega^2 - \lambda^2}{2\omega}} dq \int_{\frac{\omega(\omega - 2q) - \lambda^2}{2\omega}}^{\frac{\omega(\omega - 2q) - \lambda^2}{2(\omega - 2q)}} dr \phi(q, r, \omega - q - r)$$

$$= \int_{\frac{\lambda^2}{2\omega}}^{\frac{\omega}{2}} dq \int_{\frac{\lambda^2}{4q}}^{\frac{\omega(\omega-2q)+\lambda^2}{2\omega}} dr \phi\left(\frac{\omega}{2} - q, \frac{\omega}{2} - r, q + r\right), \quad (\text{A.27})$$

$$\begin{aligned} \text{(ii)} \quad & \int_{\frac{\omega}{2}}^{\infty} dq \int_{\frac{\omega(2q-\omega)+\lambda^2}{2(2q-\omega)}}^{\infty} dr \phi(q, r, -\omega + q + r) \\ &= \int_0^{\infty} dq \int_{\frac{\lambda^2}{4q}}^{\infty} dr \phi\left(\frac{\omega}{2} + q, \frac{\omega}{2} + r, q + r\right), \end{aligned} \quad (\text{A.28})$$

$$\begin{aligned} \text{(iii)} \quad & \int_0^{\infty} dq \int_{\frac{\omega(\omega+2q)-\lambda^2}{2(\omega+2q)}}^{\frac{\omega(\omega+2q)-\lambda^2}{2\omega}} dr \phi(q, r, \omega + q - r) \\ &= \int_{\frac{\omega}{2}}^{\infty} dq \int_{-\frac{\lambda^2}{4q}}^{\frac{\omega(2q-\omega)-\lambda^2}{2\omega}} dr \phi\left(q - \frac{\omega}{2}, \frac{\omega}{2} + r, q - r\right). \end{aligned} \quad (\text{A.29})$$

These are illustrated in fig. 10. The expression of eq. (A.20) becomes

$$\begin{aligned} \rho_{\mathcal{I}_j}^{(\text{ps})}(\omega) &= \frac{\omega^4}{2(4\pi)^3} (e^\omega - 1) \left\{ \right. \\ \text{(i)} \quad & - \int_{\frac{\lambda^2}{2\omega}}^{\frac{\omega}{2}} dq \int_{\frac{\lambda^2}{4q}}^{\frac{\omega(\omega-2q)+\lambda^2}{2\omega}} dr \mathbb{P}\left(\frac{1}{qr}\right) n_{\frac{\omega}{2}-q} n_{\frac{\omega}{2}-r} n_{q+r} \\ \text{(ii)} \quad & - \int_0^{\infty} dq \int_{\frac{\lambda^2}{4q}}^{\infty} dr \mathbb{P}\left(\frac{1}{qr}\right) n_{\frac{\omega}{2}+q} n_{\frac{\omega}{2}+r} n_{q+r} e^{q+r} \\ \text{(iii)} \quad & + \int_{\frac{\omega}{2}}^{\infty} dq \int_{-\frac{\lambda^2}{4q}}^{\frac{\omega(2q-\omega)-\lambda^2}{2\omega}} dr \mathbb{P}\left(\frac{2}{qr}\right) n_{q-\frac{\omega}{2}} n_{\frac{\omega}{2}+r} n_{q-r} e^{q-\frac{\omega}{2}} \left. \right\}. \end{aligned} \quad (\text{A.30})$$

Subsequently we can reflect the range $q < r$ to $q > r$ in cases (i) and (ii), as indicated in fig. 10; then the integrations only start at $q = \lambda/2$, and explicit divergences are averted.

So far we have made no approximations concerning λ in the phase space integrals (apart from assuming that $\lambda < \omega$), but now we again want to send $\lambda \rightarrow 0$. As mentioned this is only possible once the factorized and phase space parts are added together, but within each part we can extract the “asymptotic” behaviour in this limit. To this end we wish to re-arrange the integrations in a form in which a simple divergent part can be computed analytically and a more complicated but finite part is left over for numerical evaluation.

The divergent parts to be subtracted have two separate origins: they are either related to a prefactor $1/r$, or to the infrared divergent Bose distribution n_{q+r} (or $n_{q-\frac{\omega}{2}}$ for the case (iii)). To handle n_{q+r} we subtract a term of the form $\alpha n_{q+r}/(qr)$, where α is the “residue” of this structure at the origin; the subtracted term can be integrated analytically, by changing variables from (q, r) to $(x, y) \equiv (q + r, q - r)$, and carrying out the subsequent integration over y . To handle any remaining $1/r$ -divergences, we subtract a term $\beta(q)/(qr)$, where $\beta(q)$ is the “residue” at $r = 0$. After these subtractions the remainder integral must remain finite

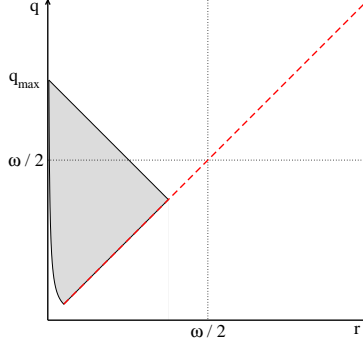


Figure 11: An integration range for which a substitution of the type in eq. (A.31) can be carried out.

even in the limit $\lambda \rightarrow 0$. (It is not clear to us whether these subtractions are the unique or most elegant ones but they do achieve their goal.)

As far as the mentioned change of integration variables goes it can be carried out, in a range illustrated in fig. 11, as

$$\begin{aligned} \mathcal{I} &\equiv \int_{\frac{\lambda}{2}}^{q_{\max}} \frac{dq}{q} \int_{\frac{\lambda^2}{4q}}^{r_{\max}(q)} \frac{dr}{r} \phi(q+r, q-r) \\ &= \int_{\lambda}^{x_{\max}} \frac{dx}{x} \int_0^{\sqrt{x^2 - \lambda^2}} dy \left(\frac{1}{x+y} + \frac{1}{x-y} \right) \phi(x, y) \Big|_{x_{\max}=q_{\max}+r_{\max}(q_{\max})} . \end{aligned} \quad (\text{A.31})$$

The upper limit of the y -integration was obtained by eliminating q in favour of x from the parametrization of the left boundary of the shaded region in fig. 11: $(x, y) = (q + \frac{\lambda^2}{4q}, q - \frac{\lambda^2}{4q})$. If the function ϕ does not depend on $y = q - r$, we finally obtain

$$\mathcal{I} = \int_{\lambda}^{x_{\max}} \frac{dx}{x} \ln \left| \frac{x + \sqrt{x^2 - \lambda^2}}{x - \sqrt{x^2 - \lambda^2}} \right| \phi(x) , \quad (\text{A.32})$$

and subsequently rename x to be q .

Let us illustrate this procedure first with the case (ii) of eq. (A.30), which is the simplest. Symmetrizing in $q \leftrightarrow r$, we cancel the $1/2$ from the prefactor (the q -integration then starts at $\lambda/2$). From the integrand, $n_{\frac{\omega}{2}+q} n_{\frac{\omega}{2}+r} n_{q+r} e^{q+r} (1 - e^{\omega})$, we first subtract n_{q+r} together with its residue, i.e.

$$n_{\frac{\omega}{2}}^2 n_{q+r} (1 - e^{\omega}) = -(1 + 2n_{\frac{\omega}{2}}) n_{q+r} . \quad (\text{A.33})$$

The value of the remainder at $r = 0$ can be simplified into

$$n_{\frac{\omega}{2}+q} n_{\frac{\omega}{2}} n_q e^q (1 - e^{\omega}) + (1 + 2n_{\frac{\omega}{2}}) n_q = (1 + 2n_{\frac{\omega}{2}}) n_{q+\frac{\omega}{2}} . \quad (\text{A.34})$$

When both eq. (A.33) and eq. (A.34) are subtracted, the final remainder simplifies tremendously, obtaining a form which vanishes for $r \rightarrow 0$ even at small q . In total, this “partitioning”

amounts to the identity

$$n_{\frac{\omega}{2}+q} n_{\frac{\omega}{2}+r} n_{q+r} e^{q+r} (1 - e^\omega) = (1 + 2n_{\frac{\omega}{2}}) \left[-n_{q+r} + n_{q+\frac{\omega}{2}} - (1 + n_{q+\frac{\omega}{2}}) \frac{n_{q+r} n_{r+\frac{\omega}{2}}}{n_r^2} \right]. \quad (\text{A.35})$$

The first term of eq. (A.35) can now be integrated with the help of eq. (A.32). The second term is independent of r so that the principal value integral in eq. (A.30) evaluates to

$$\int_{\frac{\lambda^2}{4q}}^q dr \, \mathbb{P}\left(\frac{1}{r}\right) = \ln \left| \frac{4q^2}{\lambda^2} \right|. \quad (\text{A.36})$$

As mentioned the integral over the third term remains finite when sending $\lambda \rightarrow 0$. Thereby the whole expression becomes

$$\begin{aligned} \rho_{I_j, \text{(ii)}}^{(\text{ps})}(\omega) \approx & \frac{\omega^4}{(4\pi)^3} (1 + 2n_{\frac{\omega}{2}}) \left\{ \int_{\lambda}^{\infty} \frac{dq}{q} n_q \ln \left| \frac{q - \sqrt{q^2 - \lambda^2}}{q + \sqrt{q^2 - \lambda^2}} \right| \right. \\ & \left. + \int_{\frac{\lambda}{2}}^{\infty} \frac{dq}{q} n_{q+\frac{\omega}{2}} \ln \left| \frac{4q^2}{\lambda^2} \right| - \int_0^{\infty} \frac{dq}{q} (1 + n_{q+\frac{\omega}{2}}) \int_0^q \frac{dr}{r} \frac{n_{q+r} n_{r+\frac{\omega}{2}}}{n_r^2} \right\}, \quad (\text{A.37}) \end{aligned}$$

where “ \approx ” is a reminder of having set $\lambda \rightarrow 0$ whenever possible.

As far as case (iii) goes, the only Bose distribution which diverges at the edges of the integration range is $n_{q-\frac{\omega}{2}}$. Cancelling the factor 2 against $1/2$ in the prefactor of eq. (A.30), the residue at $q = \frac{\omega}{2}, r = 0$ can be rewritten as

$$n_{q-\frac{\omega}{2}} n_{\frac{\omega}{2}}^2 (e^\omega - 1) = n_{q-\frac{\omega}{2}} (1 + 2n_{\frac{\omega}{2}}). \quad (\text{A.38})$$

When this term is subtracted and r is set to zero, we get

$$n_{q-\frac{\omega}{2}} n_{\frac{\omega}{2}} n_q e^{q-\frac{\omega}{2}} (e^\omega - 1) - n_{q-\frac{\omega}{2}} (1 + 2n_{\frac{\omega}{2}}) = -n_q (1 + 2n_{\frac{\omega}{2}}). \quad (\text{A.39})$$

Subtracting both terms, we are left over with an integrable remainder; in total,

$$n_{q-\frac{\omega}{2}} n_{\frac{\omega}{2}+r} n_{q-r} e^{q-\frac{\omega}{2}} (e^\omega - 1) = (1 + 2n_{\frac{\omega}{2}}) \left[n_{q-\frac{\omega}{2}} - n_q - n_{q-\frac{\omega}{2}} \frac{(1 + n_{q-r})(n_q - n_{r+\frac{\omega}{2}})}{n_r n_{-\frac{\omega}{2}}} \right]. \quad (\text{A.40})$$

Now, the principle value integral over the $1/r$ pole in eq. (A.30) evaluates to

$$\int_{-\frac{\lambda^2}{4q}}^{\frac{\omega(2q-\omega)-\lambda^2}{2\omega}} dr \, \mathbb{P}\left(\frac{1}{r}\right) = \ln \left| \frac{2q[\omega(\omega - 2q) + \lambda^2]}{\omega\lambda^2} \right|. \quad (\text{A.41})$$

The whole expression then becomes

$$\begin{aligned} \rho_{\mathcal{I}_j, \text{(iii)}}^{(\text{ps})}(\omega) \approx & \frac{\omega^4}{(4\pi)^3} \left(1 + 2n_{\frac{\omega}{2}}\right) \left\{ \int_{\frac{\omega}{2}}^{\infty} \frac{dq}{q} \left(n_{q-\frac{\omega}{2}} - n_q\right) \ln \left| \frac{2q[\omega(\omega - 2q) + \lambda^2]}{\omega\lambda^2} \right| \right. \\ & \left. - \int_{\frac{\omega}{2}}^{\infty} \frac{dq}{q} n_{q-\frac{\omega}{2}} \int_0^{q-\frac{\omega}{2}} \frac{dr}{r} \frac{(1 + n_{q-r})(n_q - n_{r+\frac{\omega}{2}})}{n_r n_{-\frac{\omega}{2}}} \right\}. \end{aligned} \quad (\text{A.42})$$

The case (i) is slightly more complicated than (ii) and (iii) because of the non-trivial geometry of the integration range (cf. fig. 10); nevertheless, the philosophy remains the same. Cancelling the factor $1/2$ in eq. (A.30) for symmetrization, the residue of n_{q+r} at $q = r = 0$ can be written as

$$n_{\frac{\omega}{2}}^2 n_{q+r} (1 - e^\omega) = -(1 + 2n_{\frac{\omega}{2}}) n_{q+r}. \quad (\text{A.43})$$

It will be convenient to also subtract a “vacuum” term in connection with this one, by replacing n_{q+r} by $1 + n_{q+r}$. When this structure is subtracted and we set $r \rightarrow 0$, we get the second subtraction, which can be simplified into $-(1 + 2n_{\frac{\omega}{2}}) n_{\frac{\omega}{2}-q}$. The final remainder is again a function which vanishes as $r \rightarrow 0$. The full partitioning can be expressed as

$$n_{\frac{\omega}{2}-q} n_{\frac{\omega}{2}-r} n_{q+r} (1 - e^\omega) = -(1 + 2n_{\frac{\omega}{2}}) \left[1 + n_{q+r} + n_{\frac{\omega}{2}-q} + (1 + n_{\frac{\omega}{2}-r}) \frac{n_{q+r} n_{\frac{\omega}{2}-q}}{n_r^2} \right]. \quad (\text{A.44})$$

The first two terms of eq. (A.44), which only depend on $q+r$, are integrated as in eq. (A.32). The r -integrals over the third term evaluate to

$$\int_{\frac{\lambda^2}{4q}}^q dr \, \mathbb{P}\left(\frac{1}{r}\right) = \ln \left| \frac{4q^2}{\lambda^2} \right|, \quad (\text{A.45})$$

$$\int_{\frac{\lambda^2}{4q}}^{\frac{\omega(\omega-2q)+\lambda^2}{2\omega}} dr \, \mathbb{P}\left(\frac{1}{r}\right) = \ln \left| \frac{2q[\omega(\omega - 2q) + \lambda^2]}{\omega\lambda^2} \right|. \quad (\text{A.46})$$

Setting again $\lambda \rightarrow 0$ whenever possible, the whole expression becomes

$$\begin{aligned} \rho_{\mathcal{I}_j, \text{(i)}}^{(\text{ps})}(\omega) \approx & \frac{\omega^4}{(4\pi)^3} \left(1 + 2n_{\frac{\omega}{2}}\right) \left\{ \int_{\lambda}^{\frac{\omega}{2}} \frac{dq}{q} (1 + n_q) \ln \left| \frac{q - \sqrt{q^2 - \lambda^2}}{q + \sqrt{q^2 - \lambda^2}} \right| \right. \\ & + \int_{\frac{\lambda}{2}}^{\frac{\omega}{2}} \frac{dq}{q} n_{\frac{\omega}{2}-q} \ln \left| \frac{\lambda^2}{4q^2} \right| + \int_{\frac{\omega}{4}}^{\frac{\omega}{2}} \frac{dq}{q} n_{\frac{\omega}{2}-q} \ln \left| \frac{\omega\lambda^2}{2q[\omega(\omega - 2q) + \lambda^2]} \right| \\ & \left. - \int_0^{\frac{\omega}{2}} \frac{dq}{q} n_{\frac{\omega}{2}-q} \int_0^{\frac{\omega}{4}-|q-\frac{\omega}{4}|} \frac{dr}{r} \frac{n_{q+r} (1 + n_{\frac{\omega}{2}-r})}{n_r^2} \right\}. \end{aligned} \quad (\text{A.47})$$

A.5. Collecting the pieces

Combining now everything together, the first observation is that the factorized powerlike term, eq. (A.12), and the very first term from eq. (A.47), with unity in the numerator, cancel

against each other. This implies, in particular, that $\rho_{\mathcal{I}_j}$ vanishes at $T = 0$, because these are the only terms which are not (necessarily) proportional to a Bose distribution.

It is more cumbersome to deal with the 1-dimensional integrals containing n_q , $n_{|q-\frac{\omega}{2}|}$, and $n_{q+\frac{\omega}{2}}$. Summing together the terms from eqs. (A.14), (A.37), (A.42) and (A.47), we obtain

$$\begin{aligned}
\rho_{\mathcal{I}_j}^{(1d)}(\omega) &\equiv \frac{\omega^4}{(4\pi)^3} (1 + 2n_{\frac{\omega}{2}}) \left\{ \right. \\
&\quad (a) \quad \int_{\lambda}^{\frac{\omega}{2}} \frac{dq}{q} n_q \ln \left| \frac{(q + \sqrt{q^2 - \lambda^2})^2 - \frac{\lambda^4}{\omega^2}}{(q - \sqrt{q^2 - \lambda^2})^2 - \frac{\lambda^4}{\omega^2}} \right| \left| \frac{q - \sqrt{q^2 - \lambda^2}}{q + \sqrt{q^2 - \lambda^2}} \right|^2 \\
&\quad (b) \quad + \int_{\frac{\omega}{2}}^{\infty} \frac{dq}{q} n_q \ln \left| \frac{(q + \sqrt{q^2 - \lambda^2})^2 - \frac{\lambda^4}{\omega^2}}{(q - \sqrt{q^2 - \lambda^2})^2 - \frac{\lambda^4}{\omega^2}} \right| \left| \frac{q - \sqrt{q^2 - \lambda^2}}{q + \sqrt{q^2 - \lambda^2}} \right| \left| \frac{\omega \lambda^2}{2q[\omega(\omega - 2q) + \lambda^2]} \right| \\
&\quad (c) \quad + \int_0^{\infty} dq n_q \mathbb{P} \left[\frac{1}{q + \frac{\omega}{2}} \ln \left| \frac{\lambda^2}{2q\omega - \lambda^2} \right| + \frac{1}{q - \frac{\omega}{2}} \ln \left| \frac{\lambda^2}{2q\omega + \lambda^2} \right| \right] \\
&\quad (d) \quad + \int_{\frac{\lambda}{2}}^{\frac{\omega}{4}} \frac{dq}{q} \left[n_{\frac{\omega}{2}+q} \ln \left| \frac{4q^2}{\lambda^2} \right| + n_{\frac{\omega}{2}-q} \ln \left| \frac{\lambda^2}{4q^2} \right| \right] \\
&\quad (e) \quad + \int_{\frac{\omega}{4}}^{\frac{\omega}{2}} \frac{dq}{q} \left[n_{q+\frac{\omega}{2}} \ln \left| \frac{4q^2}{\lambda^2} \right| + n_{\frac{\omega}{2}-q} \ln \left| \frac{\omega \lambda^2}{2q[\omega(\omega - 2q) + \lambda^2]} \right| \right] \\
&\quad (f) \quad + \int_{\frac{\omega}{2}}^{\infty} \frac{dq}{q} \left[n_{q+\frac{\omega}{2}} \ln \left| \frac{4q^2}{\lambda^2} \right| + n_{q-\frac{\omega}{2}} \ln \left| \frac{2q[\omega(\omega - 2q) + \lambda^2]}{\omega \lambda^2} \right| \right] \left. \right\}. \tag{A.48}
\end{aligned}$$

The subsequent step is to take the limit $\lambda \rightarrow 0$. The most subtle in this respect are the cases (a) and (b). The various structures appearing there can be simplified as

$$\begin{aligned}
\left| \frac{q + \sqrt{q^2 - \lambda^2} + \frac{\lambda^2}{\omega}}{q - \sqrt{q^2 - \lambda^2} + \frac{\lambda^2}{\omega}} \right| \left| \frac{q - \sqrt{q^2 - \lambda^2}}{q + \sqrt{q^2 - \lambda^2}} \right| &= \left| \frac{(q + \sqrt{q^2 - \lambda^2} + \frac{\lambda^2}{\omega})^2}{(q + \frac{\lambda^2}{\omega})^2 - (q^2 - \lambda^2)} \right| \left| \frac{q^2 - (q^2 - \lambda^2)}{(q + \sqrt{q^2 - \lambda^2})^2} \right| \\
&= \frac{\lambda^2}{\frac{2q\lambda^2}{\omega} + \frac{\lambda^4}{\omega^2} + \lambda^2} \left(\frac{q + \sqrt{q^2 - \lambda^2} + \frac{\lambda^2}{\omega}}{q + \sqrt{q^2 - \lambda^2}} \right)^2 \stackrel{\lambda \ll \omega}{\approx} \frac{\omega^2}{\omega(\omega + 2q) + \lambda^2}, \tag{A.49}
\end{aligned}$$

$$\begin{aligned}
\left| \frac{q + \sqrt{q^2 - \lambda^2} - \frac{\lambda^2}{\omega}}{q - \sqrt{q^2 - \lambda^2} - \frac{\lambda^2}{\omega}} \right| \left| \frac{q - \sqrt{q^2 - \lambda^2}}{q + \sqrt{q^2 - \lambda^2}} \right| &= \left| \frac{(q + \sqrt{q^2 - \lambda^2} - \frac{\lambda^2}{\omega})^2}{(q - \frac{\lambda^2}{\omega})^2 - (q^2 - \lambda^2)} \right| \left| \frac{q^2 - (q^2 - \lambda^2)}{(q + \sqrt{q^2 - \lambda^2})^2} \right| \\
&= \left| \frac{\lambda^2}{-\frac{2q\lambda^2}{\omega} + \frac{\lambda^4}{\omega^2} + \lambda^2} \right| \left(\frac{q + \sqrt{q^2 - \lambda^2} - \frac{\lambda^2}{\omega}}{q + \sqrt{q^2 - \lambda^2}} \right)^2 \stackrel{\lambda \ll \omega}{\approx} \left| \frac{\omega^2}{\omega(\omega - 2q) + \lambda^2} \right|, \tag{A.50}
\end{aligned}$$

$$\begin{aligned}
\left| \frac{q + \sqrt{q^2 - \lambda^2} - \frac{\lambda^2}{\omega}}{q - \sqrt{q^2 - \lambda^2} - \frac{\lambda^2}{\omega}} \right| \left| \frac{\omega \lambda^2}{2q[\omega(\omega - 2q) + \lambda^2]} \right| &= \left| \frac{(q + \sqrt{q^2 - \lambda^2} - \frac{\lambda^2}{\omega})^2}{(q - \frac{\lambda^2}{\omega})^2 - (q^2 - \lambda^2)} \right| \left| \frac{\omega \lambda^2}{2q[\omega(\omega - 2q) + \lambda^2]} \right| \\
&= \left| \frac{\frac{\omega \lambda^2}{2q}}{-\frac{2q\lambda^2}{\omega} + \frac{\lambda^4}{\omega^2} + \lambda^2} \right| \left| \frac{(q + \sqrt{q^2 - \lambda^2} - \frac{\lambda^2}{\omega})^2}{\omega(\omega - 2q) + \lambda^2} \right| \stackrel{\lambda \ll \omega}{\approx} \frac{2q\omega}{(\omega - 2q + \frac{\lambda^2}{\omega})^2}. \tag{A.51}
\end{aligned}$$

These then lead to

$$(a) \approx \int_0^{\frac{\omega}{2}} \frac{dq}{q} n_q \ln \frac{\omega^2}{(\omega - 2q)(\omega + 2q)}, \quad (A.52)$$

$$(b) \approx \int_{\frac{\omega}{2}}^{\infty} \frac{dq}{q} n_q \ln \frac{2\omega^2 q}{(\omega - 2q)^2(\omega + 2q)}. \quad (A.53)$$

In case (d) we observe that the lower limit of the integration can safely be set to zero *in the sum*. With this knowledge in mind, the left-most terms of (d), (e) and (f) can be summed together and the integration variable can be shifted as $q \rightarrow -\frac{\omega}{2} + q$, to obtain

$$(d+e+f)_{\text{left}} \approx \int_{\frac{\omega}{2}}^{\infty} dq n_q \left[\frac{1}{q - \frac{\omega}{2}} \ln \frac{(\omega - 2q)^2}{\lambda^2} \right]. \quad (A.54)$$

In the right-most terms of (d) and (e) we can shift $q \rightarrow \frac{\omega}{2} - q$, and in the right-most term of (f), we can shift $q \rightarrow \frac{\omega}{2} + q$, thereby obtaining

$$(d+e)_{\text{right}} \approx \int_0^{\frac{\omega}{2}} dq n_q \left[\frac{\theta(\frac{\omega}{4} - q)}{q - \frac{\omega}{2}} \ln \left| \frac{(\omega - 2q)(2\omega q + \lambda^2)}{\omega \lambda^2} \right| + \frac{\theta(q - \frac{\omega}{4})}{q - \frac{\omega}{2}} \ln \left| \frac{(\omega - 2q)^2}{\lambda^2} \right| \right], \quad (A.55)$$

$$(f)_{\text{right}} \approx \int_0^{\infty} dq n_q \left[\frac{1}{q + \frac{\omega}{2}} \ln \left| \frac{(\omega + 2q)(2\omega q - \lambda^2)}{\omega \lambda^2} \right| \right]. \quad (A.56)$$

Summing all of these together with (c), the λ 's cancel or become harmless, and the regularization observed in (d), momentarily implicit in eq. (A.54), is rediscovered as a principle value integral over $1/(q - \frac{\omega}{2})$. We thereby obtain the first three rows of eq. (A.57) below.

A.6. Final expression

Putting together the 2-dimensional integrals from eqs. (A.37), (A.42) and (A.47), and the 1-dimensional integrals from sec. A.5, we obtain a convergent expression for $\rho_{\mathcal{I}_j}$ at $\lambda \rightarrow 0$:

$$\begin{aligned} & \frac{(4\pi)^3 \rho_{\mathcal{I}_j}(\omega)}{\omega^4(1 + 2n_{\frac{\omega}{2}})} = \\ & \int_0^{\frac{\omega}{4}} dq n_q \left[\left(\frac{1}{q - \frac{\omega}{2}} - \frac{1}{q} \right) \ln \left(1 - \frac{2q}{\omega} \right) - \frac{\frac{\omega}{2}}{q(q + \frac{\omega}{2})} \ln \left(1 + \frac{2q}{\omega} \right) \right] \\ & + \int_{\frac{\omega}{4}}^{\frac{\omega}{2}} dq n_q \left[\left(\frac{2}{q - \frac{\omega}{2}} - \frac{1}{q} \right) \ln \left(1 - \frac{2q}{\omega} \right) - \frac{\frac{\omega}{2}}{q(q + \frac{\omega}{2})} \ln \left(1 + \frac{2q}{\omega} \right) - \frac{1}{q - \frac{\omega}{2}} \ln \left(\frac{2q}{\omega} \right) \right] \\ & + \int_{\frac{\omega}{2}}^{\infty} dq n_q \left[\left(\frac{2}{q - \frac{\omega}{2}} - \frac{2}{q} \right) \ln \left(\frac{2q}{\omega} - 1 \right) - \frac{\frac{\omega}{2}}{q(q + \frac{\omega}{2})} \ln \left(1 + \frac{2q}{\omega} \right) + \left(\frac{1}{q} - \frac{1}{q - \frac{\omega}{2}} \right) \ln \left(\frac{2q}{\omega} \right) \right] \\ & + \int_0^{\frac{\omega}{2}} dq \int_0^{\frac{\omega}{4} - |q - \frac{\omega}{4}|} dr \left(-\frac{1}{qr} \right) \frac{n_{\frac{\omega}{2} - q} n_{q+r} (1 + n_{\frac{\omega}{2} - r})}{n_r^2} \end{aligned}$$

$$\begin{aligned}
& + \int_{\frac{\omega}{2}}^{\infty} dq \int_0^{q-\frac{\omega}{2}} dr \left(-\frac{1}{qr} \right) \frac{n_{q-\frac{\omega}{2}}(1+n_{q-r})(n_q - n_{r+\frac{\omega}{2}})}{n_r n_{-\frac{\omega}{2}}} \\
& + \int_0^{\infty} dq \int_0^q dr \left(-\frac{1}{qr} \right) \frac{(1+n_{q+\frac{\omega}{2}})n_{q+r}n_{r+\frac{\omega}{2}}}{n_r^2} + \mathcal{O}(\lambda \ln \lambda) .
\end{aligned} \tag{A.57}$$

For numerical evaluation, it is helpful to combine the first terms of the 2nd and 3rd rows into

$$\int_{\frac{\omega}{4}}^{\infty} dq \mathbb{P}\left(\frac{2n_q}{q-\frac{\omega}{2}}\right) \ln\left|1 - \frac{2q}{\omega}\right| = \int_0^{\frac{\omega}{4}} \frac{dq}{q} \left(n_{\frac{\omega}{2}+q} - n_{\frac{\omega}{2}-q}\right) \ln \frac{4q^2}{\omega^2} + \int_{\frac{3\omega}{4}}^{\infty} dq \frac{2n_q}{q-\frac{\omega}{2}} \ln\left(\frac{2q}{\omega} - 1\right) . \tag{A.58}$$

A.7. Ultraviolet asymptotics

As a check of the result obtained, it is useful to verify that it reproduces the known behaviour in the limit $\omega \gg \pi T$. From ref. [36], eq. (A.32), we recall that asymptotically \mathcal{I}_j reads

$$\begin{aligned}
\mathcal{I}_j &= \frac{\mathcal{S}_1(1-\mathcal{S}_2)}{\epsilon^3(1-2\epsilon)} - \frac{2(3+\epsilon)\mathcal{S}_3}{\epsilon} - \frac{2(5+\epsilon)(10+5\epsilon+\epsilon^2)\mathcal{S}_5}{3\epsilon} \\
&+ \left[\frac{2(3+\epsilon)}{2-\epsilon} + \frac{20(1-\epsilon)^2}{\epsilon(2-\epsilon)P^2} \left(\frac{p^2}{3-2\epsilon} - p_n^2 \right) \right] \mathcal{S}_6 + \mathcal{O}\left(\frac{1}{P^2}\right) ,
\end{aligned} \tag{A.59}$$

with

$$\mathcal{S}_1 \equiv \frac{P^{4-4\epsilon}}{(4\pi)^{4-2\epsilon}} \frac{\Gamma^2(1+\epsilon)\Gamma^4(1-\epsilon)}{\Gamma^2(1-2\epsilon)} , \tag{A.60}$$

$$\mathcal{S}_2 \equiv \frac{\Gamma(1+2\epsilon)\Gamma^2(1-2\epsilon)}{\Gamma(1-3\epsilon)\Gamma^2(1+\epsilon)\Gamma(1-\epsilon)} , \tag{A.61}$$

$$\mathcal{S}_3 \equiv \frac{P^{2-2\epsilon}}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \int_{\mathbf{q}} \frac{n_{\mathbf{B}}(q)}{q} , \tag{A.62}$$

$$\mathcal{S}_4 \equiv \frac{(1-2\epsilon)P^2}{2} \int_{\mathbf{q}, \mathbf{r}} \frac{n_{\mathbf{B}}(q)}{q} \frac{n_{\mathbf{B}}(r)}{r^3} , \tag{A.63}$$

$$\mathcal{S}_5 \equiv \frac{P^{-2\epsilon}}{(4\pi)^{2-\epsilon}} \frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)} \frac{1}{P^2} \left(\frac{p^2}{3-2\epsilon} - p_n^2 \right) \int_{\mathbf{q}} q n_{\mathbf{B}}(q) , \tag{A.64}$$

$$\mathcal{S}_6 \equiv \int_{\mathbf{q}, \mathbf{r}} \frac{n_{\mathbf{B}}(q)}{q} \frac{n_{\mathbf{B}}(r)}{r} . \tag{A.65}$$

Setting $P = (p_n, \mathbf{0})$, analytically continuing to Minkowski signature, $p_n \rightarrow -i[\omega + i0^+]$, and extracting the spectral function according to eq. (A.2), yields

$$\rho_{\mathcal{S}_1} = 2\pi\epsilon \frac{\omega^4 \mu^{-4\epsilon}}{(4\pi)^4} \left(1 + 2\epsilon \ln \frac{\bar{\mu}^2}{\omega^2} \right) + \mathcal{O}(\epsilon^3) , \tag{A.66}$$

$$\rho_{\mathcal{S}_2} = 1 + \mathcal{O}(\epsilon^3) , \tag{A.67}$$

$$\rho_{\mathcal{S}_3} = -\pi\epsilon \frac{\omega^2 \mu^{-2\epsilon}}{(4\pi)^2} \frac{T^2}{12} + \mathcal{O}(\epsilon^2) , \quad (\text{A.68})$$

$$\rho_{\mathcal{S}_4} = 0 , \quad (\text{A.69})$$

$$\rho_{\mathcal{S}_5} = -\pi\epsilon \frac{\mu^{-2\epsilon}}{(4\pi)^2} \frac{\pi^2 T^4}{30} + \mathcal{O}(\epsilon^2) , \quad (\text{A.70})$$

$$\rho_{\mathcal{S}_6} = 0 , \quad (\text{A.71})$$

where $\mu^2 = \bar{\mu}^2 e^{\gamma_E}/4\pi$. Inserting these into eq. (A.59) we notice that the temperature-independent part is of $\mathcal{O}(\epsilon)$ and therefore vanishes in the continuum limit, just as we found at the beginning of sec. A.5. The remaining asymptotics reads

$$\rho_{\mathcal{I}_j}^{\text{OPE}}(\omega) = \frac{\omega^2 T^2}{32\pi} + \frac{5\pi T^4}{72} + \mathcal{O}\left(\frac{T^6}{\omega^2}\right) . \quad (\text{A.72})$$

It is easy to check numerically that eq. (A.57) indeed approaches this form at $\omega \gg \pi T$.

B. Main results for the other cases

Below we list the spectral functions corresponding to the master sum-integrals defined in eqs. (3.4)–(3.14). Like in appendix A, all terms containing structures of the type $\omega^n \delta(\omega)$ are omitted from the outset although, as discussed in sec. 5.6, under specific circumstances such terms may conspire to give a “zero-mode” contribution; we do not believe this to be the case for the observables and perturbative order considered in the present investigation.

B.1. $\rho_{\mathcal{J}_a}$

The sum-integral \mathcal{J}_a is defined as

$$\mathcal{J}_a \equiv \not\!\!\!\int_Q \frac{P^2}{Q^2} . \quad (\text{B.1})$$

Since the dependence on p_n is polynomial, there is no cut, and the corresponding spectral function vanishes:

$$\rho_{\mathcal{J}_a}(\omega) = 0 . \quad (\text{B.2})$$

B.2. $\rho_{\mathcal{J}_b}$

The sum-integral \mathcal{J}_b is defined as

$$\mathcal{J}_b \equiv \not\!\!\!\int_Q \frac{P^4}{Q^2(Q-P)^2} . \quad (\text{B.3})$$

Carrying out the Matsubara sum, taking the cut, and setting subsequently $\mathbf{p} \rightarrow \mathbf{0}$, we get

$$\rho_{\mathcal{J}_b}(\omega) = \int_{\mathbf{q}} \frac{\omega^4 \pi}{4q^2} \left[\delta(\omega - 2q) - \delta(\omega + 2q) \right] (1 + 2n_q) . \quad (\text{B.4})$$

Making use of the usual dimensionally regularized integration measure, the integral over the Dirac- δ yields

$$\int_{\mathbf{q}} \pi \delta(\omega - 2q) = \frac{\omega^2 \mu^{-2\epsilon}}{16\pi} \left[1 + \epsilon \left(\ln \frac{\bar{\mu}^2}{\omega^2} + 2 \right) + \mathcal{O}(\epsilon^2) \right] , \quad (\text{B.5})$$

and the complete expression reads (for $\omega > 0$)

$$\rho_{\mathcal{J}_b}(\omega) = \frac{\omega^4 \mu^{-2\epsilon}}{16\pi} (1 + 2n_{\frac{\omega}{2}}) \left[1 + \epsilon \left(\ln \frac{\bar{\mu}^2}{\omega^2} + 2 \right) + \mathcal{O}(\epsilon^2) \right] . \quad (\text{B.6})$$

Thermal effects are seen to be exponentially suppressed here, so we nicely match the asymptotic OPE-result as determined in ref. [36],

$$\rho_{\mathcal{J}_b}^{\text{OPE}}(\omega) = \frac{\omega^4 \mu^{-2\epsilon}}{16\pi} \left[1 + \epsilon \left(\ln \frac{\bar{\mu}^2}{\omega^2} + 2 \right) \right] + \mathcal{O}\left(\frac{T^6}{\omega^2}\right) . \quad (\text{B.7})$$

B.3. $\rho_{\mathcal{I}_a}$

The sum-integral \mathcal{I}_a is defined as

$$\mathcal{I}_a \equiv \not\!\!\!\int_{Q,R} \frac{1}{Q^2 R^2} . \quad (\text{B.8})$$

Since there is no dependence on P , the corresponding spectral function vanishes:

$$\rho_{\mathcal{I}_a}(\omega) = 0 . \quad (\text{B.9})$$

B.4. $\rho_{\mathcal{I}_b}$

The sum-integral \mathcal{I}_b is defined as

$$\mathcal{I}_b \equiv \not\!\!\!\int_{Q,R} \frac{P^2}{Q^2 R^2 (R - P)^2} . \quad (\text{B.10})$$

This factorizes into the same form as \mathcal{J}_b in eq. (B.3), multiplied by the well-known

$$\not\!\!\!\int_Q \frac{1}{Q^2} = \frac{T^2}{12} + \mathcal{O}(\epsilon) . \quad (\text{B.11})$$

Within the 2-loop contributions we only need to go to $\mathcal{O}(\epsilon^0)$, so the result reads

$$\rho_{\mathcal{I}_b}(\omega) = -\frac{\omega^2 T^2}{192\pi} (1 + 2n_{\frac{\omega}{2}}) . \quad (\text{B.12})$$

For $\omega \gg \pi T$ this agrees with the asymptotic OPE-form from ref. [36],

$$\rho_{\mathcal{I}_b}^{\text{OPE}}(\omega) = -\frac{\omega^2 T^2}{192\pi} + \mathcal{O}\left(\frac{T^6}{\omega^2}\right) . \quad (\text{B.13})$$

B.5. $\rho_{\mathcal{I}_c}$

The sum-integral \mathcal{I}_c is defined as

$$\mathcal{I}_c \equiv \not\!\!\!\int_{Q,R} \frac{P^2}{Q^2 R^4} . \quad (\text{B.14})$$

Since the dependence on p_n is polynomial, there is no cut, and the corresponding spectral function vanishes:

$$\rho_{\mathcal{I}_c}(\omega) = 0 . \quad (\text{B.15})$$

B.6. $\rho_{\mathcal{I}_d}$

The sum-integral \mathcal{I}_d is defined as

$$\mathcal{I}_d \equiv \oint_{Q,R} \frac{P^4}{Q^2 R^4 (R-P)^2} . \quad (\text{B.16})$$

This factorizes into the tadpole in eq. (B.11), times a 1-loop sum-integral. For the latter, a possible trick is to write

$$\begin{aligned} \oint_R \frac{1}{R^4 (R-P)^2} &= \lim_{\lambda \rightarrow 0} \frac{1}{2} \oint_R \left\{ \frac{1}{[R^2 + \lambda^2]^2 [(R-P)^2 + \lambda^2]} + \frac{1}{[R^2 + \lambda^2] [(R-P)^2 + \lambda^2]^2} \right\} \\ &= \lim_{\lambda \rightarrow 0} \left\{ -\frac{1}{2} \frac{d}{d\lambda^2} \oint_R \frac{1}{[R^2 + \lambda^2] [(R-P)^2 + \lambda^2]} \right\} . \end{aligned} \quad (\text{B.17})$$

The result for the Matsubara sum here (after taking the cut and setting $\mathbf{p} \rightarrow \mathbf{0}$) can be read directly from eq. (B.4),

$$\begin{aligned} \oint_R \frac{1}{[R^2 + \lambda^2] [(R-P)^2 + \lambda^2]} &\rightarrow \int_{\mathbf{r}} \frac{\pi}{4E_r^2} \left[\delta(\omega - 2E_r) - \delta(\omega + 2E_r) \right] (1 + 2n_{E_r}) \\ &= \frac{(\omega^2 - 4\lambda^2)^{\frac{1}{2}}}{16\pi\omega} (1 + 2n_{\frac{\omega}{2}}) \theta(\omega - 2\lambda) + \mathcal{O}(\epsilon) , \end{aligned} \quad (\text{B.18})$$

where $E_r^2 \equiv r^2 + \lambda^2$ and the spatial integral was performed for $\omega > 0$. Taking the derivative in eq. (B.17) and setting $\lambda \rightarrow 0$ finally yields

$$\rho_{\mathcal{I}_d}(\omega) = \frac{\omega^2 T^2}{192\pi} (1 + 2n_{\frac{\omega}{2}}) = \frac{2\omega^2}{(4\pi)^3} (1 + 2n_{\frac{\omega}{2}}) \int_0^\infty dq n_q q , \quad (\text{B.19})$$

where we have also given an integral representation, in order to facilitate comparison with the other spectral functions below. The result matches the OPE-behaviour from ref. [36],

$$\rho_{\mathcal{I}_d}^{\text{OPE}}(\omega) = \frac{\omega^2 T^2}{192\pi} + \mathcal{O}\left(\frac{T^6}{\omega^2}\right) . \quad (\text{B.20})$$

B.7. $\rho_{\mathcal{I}_e}$

The sum-integral \mathcal{I}_e is defined as

$$\mathcal{I}_e \equiv \oint_{Q,R} \frac{P^2}{Q^2 R^2 (Q-R)^2} . \quad (\text{B.21})$$

Again the result is a polynomial in p_n and has no cut; in fact even before the cut this massless sum-integral vanishes exactly in dimensional regularization in any dimension [45]. Therefore,

$$\rho_{\mathcal{I}_e}(\omega) = 0 . \quad (\text{B.22})$$

B.8. $\rho_{\mathcal{I}_f}$

The sum-integral \mathcal{I}_f is defined as

$$\mathcal{I}_f \equiv \oint_{Q,R} \frac{P^2}{Q^2[(Q-R)^2 + \lambda^2](R-P)^2} . \quad (\text{B.23})$$

After the Matsubara sums the expression for the corresponding spectral function reads

$$\begin{aligned} \rho_{\mathcal{I}_f}(\omega) = \int_{\mathbf{q},\mathbf{r}} \frac{\omega^2 \pi}{8qr E_{qr}} \Big\{ & \\ & - \left[\delta(\omega - q - r - E_{qr}) - \delta(\omega + q + r + E_{qr}) \right] \left[(1 + n_{qr})(1 + n_q + n_r) + n_q n_r \right] \\ & - \left[\delta(\omega - q - r + E_{qr}) - \delta(\omega + q + r - E_{qr}) \right] \left[n_{qr}(1 + n_q + n_r) - n_q n_r \right] \\ & - \left[\delta(\omega - q + r - E_{qr}) - \delta(\omega + q - r + E_{qr}) \right] \left[n_r(1 + n_q + n_{qr}) - n_q n_{qr} \right] \\ & - \left[\delta(\omega + q - r - E_{qr}) - \delta(\omega - q + r + E_{qr}) \right] \left[n_q(1 + n_r + n_{qr}) - n_r n_{qr} \right] \Big\} . \end{aligned} \quad (\text{B.24})$$

Evidently this sum-integral only contains a phase-space part, and the expression corresponding to eq. (A.30) becomes

$$\begin{aligned} \rho_{\mathcal{I}_f}^{(\text{ps})}(\omega) &= \frac{\omega^2}{(4\pi)^3} (1 - e^\omega) \Big\{ \\ & \text{(i)} \quad \int_{\frac{\lambda^2}{2\omega}}^{\frac{\omega}{2}} dq \int_{\frac{\lambda^2}{4q}}^{\frac{\omega(\omega-2q)+\lambda^2}{2\omega}} dr \, n_{\frac{\omega}{2}-q} n_{\frac{\omega}{2}-r} n_{q+r} \\ & \text{(ii)} \quad + \int_0^\infty dq \int_{\frac{\lambda^2}{4q}}^\infty dr \, n_{\frac{\omega}{2}+q} n_{\frac{\omega}{2}+r} n_{q+r} e^{q+r} \\ & \text{(iii)} \quad + 2 \int_{\frac{\omega}{2}}^\infty dq \int_{-\frac{\lambda^2}{4q}}^{\frac{\omega(2q-\omega)-\lambda^2}{2\omega}} dr \, n_{q-\frac{\omega}{2}} n_{\frac{\omega}{2}+r} n_{q-r} e^{q-\frac{\omega}{2}} \Big\} . \end{aligned} \quad (\text{B.25})$$

Subsequently we can reflect the range $q < r$ to $q > r$ in cases (i) and (ii). The subtractions of the potentially divergent parts are carried out like in eqs. (A.35), (A.40), (A.44); the integrals over y and r are trivial in these terms.

Once the pieces are collected together, we can again identify a “vacuum part”, a “1-dimensional integral”, and a “2-dimensional integral”. The contribution to the vacuum part from the phase space integrals reads

$$\frac{\omega^2}{(4\pi)^3} (1 + 2n_{\frac{\omega}{2}}) \left\{ - \int_{\lambda}^{\frac{\omega}{2}} dq \, q \right\} \approx - \frac{\omega^4}{8(4\pi)^3} (1 + 2n_{\frac{\omega}{2}}) . \quad (\text{B.26})$$

This yields the first row of eq. (B.27). The 1-dimensional integrals have no delicate divergences here. Setting λ to zero, we obtain

$$\begin{aligned}
& \frac{(4\pi)^3 \rho_{\mathcal{I}_f}(\omega)}{2\omega^2(1+2n_{\frac{\omega}{2}})} = -\frac{\omega^2 \mu^{-4\epsilon}}{16} \\
& - \int_0^{\frac{\omega}{4}} dq n_q \left(3q\right) - \int_{\frac{\omega}{4}}^{\frac{\omega}{2}} dq n_q \left(q + \frac{\omega}{2}\right) + \int_{\frac{\omega}{2}}^{\infty} dq n_q \left(\frac{q}{2} - \omega\right) \\
& - \int_0^{\frac{\omega}{2}} dq \int_0^{\frac{\omega}{4}-|q-\frac{\omega}{4}|} dr \frac{n_{\frac{\omega}{2}-q} n_{q+r} (1+n_{\frac{\omega}{2}-r})}{n_r^2} \\
& + \int_{\frac{\omega}{2}}^{\infty} dq \int_0^{q-\frac{\omega}{2}} dr \frac{n_{q-\frac{\omega}{2}} (1+n_{q-r}) (n_q - n_{r+\frac{\omega}{2}})}{n_r n_{-\frac{\omega}{2}}} \\
& - \int_0^{\infty} dq \int_0^q dr \frac{(1+n_{q+\frac{\omega}{2}}) n_{q+r} n_{r+\frac{\omega}{2}}}{n_r^2} + \mathcal{O}(\lambda \ln \lambda) . \tag{B.27}
\end{aligned}$$

For $\omega \gg \pi T$ it can be checked numerically that this goes over to the known [36] ultraviolet asymptotics

$$\rho_{\mathcal{I}_f}^{\text{OPE}}(\omega) = -\frac{\omega^4 \mu^{-4\epsilon}}{8(4\pi)^3} - \frac{\omega^2 T^2}{64\pi} + \mathcal{O}\left(\frac{T^6}{\omega^2}\right) . \tag{B.28}$$

B.9. $\rho_{\mathcal{I}_g}$

The sum-integral \mathcal{I}_g is defined as

$$\mathcal{I}_g \equiv \oint_{Q,R} \frac{P^4}{Q^2(Q-P)^2 R^2(R-P)^2} . \tag{B.29}$$

After the Matsubara sums the expression for the corresponding spectral function reads

$$\begin{aligned}
\rho_{\mathcal{I}_g}(\omega) = \int_{\mathbf{q},\mathbf{r}} \frac{\omega^4 \pi}{32q^2 r^2} (1+2n_q)(1+2n_r) \Big\{ \\
\left[\delta(\omega-2r) - \delta(\omega+2r) \right] \left(\frac{1}{q-r} + \frac{1}{q+r} \right) \\
+ \left[\delta(\omega-2q) - \delta(\omega+2q) \right] \left(\frac{1}{r-q} + \frac{1}{r+q} \right) \Big\} . \tag{B.30}
\end{aligned}$$

The two terms can be combined through $q \leftrightarrow r$ if we first introduce a principle value prescription. Choosing the first one as a representative, the \mathbf{r} -integral can be carried out like in eq. (B.5), and we obtain

$$\begin{aligned}
\rho_{\mathcal{I}_g}(\omega) = \frac{\omega^4 \mu^{-2\epsilon}}{64\pi} (1+2n_{\frac{\omega}{2}}) \left[1 + \epsilon \left(\ln \frac{\bar{\mu}^2}{\omega^2} + 2 \right) \right] \times \\
\times \left\{ \int_{\mathbf{q}} \frac{1}{q^2} \mathbb{P} \left(\frac{1}{q-\frac{\omega}{2}} + \frac{1}{q+\frac{\omega}{2}} \right) + \frac{1}{\pi^2} \int_0^{\infty} dq n_q \mathbb{P} \left(\frac{1}{q-\frac{\omega}{2}} + \frac{1}{q+\frac{\omega}{2}} \right) \right\} . \tag{B.31}
\end{aligned}$$

The thermal part (with n_q) is finite as it stands, whereas the vacuum part may be transformed into a well-known four-dimensional integral:

$$\begin{aligned}
\int_{\mathbf{q}} \frac{1}{q^2} \mathbb{P} \left(\frac{1}{q - \frac{\omega}{2}} + \frac{1}{q + \frac{\omega}{2}} \right) &= 8 \int_{\mathbf{q}} \frac{1}{4q^2} \mathbb{P} \left(\frac{1}{\omega + 2q} + \frac{1}{-\omega + 2q} \right) \\
&= 8 \operatorname{Re} \left\{ \int_Q \frac{1}{Q^2(Q - R)^2} \Big|_{R=(i\omega, \mathbf{0})} \right\} \\
&= 8 \operatorname{Re} \left\{ \frac{\mu^{-2\epsilon}}{(4\pi)^2} \left[\frac{1}{\epsilon} + \ln \left(\frac{\bar{\mu}^2}{-\omega^2} \right) + 2 + \mathcal{O}(\epsilon) \right] \right\}. \quad (\text{B.32})
\end{aligned}$$

This can be verified by carrying out $\int \frac{dq_0}{2\pi}$ on the right-hand side. In total, omitting $\mathcal{O}(\epsilon)$,

$$\frac{(4\pi)^3 \rho_{\mathcal{I}_g}(\omega)}{\omega^4(1 + 2n_{\frac{\omega}{2}})} = \frac{\mu^{-4\epsilon}}{2} \left(\frac{1}{\epsilon} + 2 \ln \frac{\bar{\mu}^2}{\omega^2} + 4 \right) + \int_0^\infty dq n_q \mathbb{P} \left(\frac{1}{q - \frac{\omega}{2}} + \frac{1}{q + \frac{\omega}{2}} \right). \quad (\text{B.33})$$

For $\omega \gg \pi T$ this approaches the OPE-result [36]

$$\rho_{\mathcal{I}_g}^{\text{OPE}}(\omega) = \frac{\omega^4 \mu^{-4\epsilon}}{2(4\pi)^3} \left(\frac{1}{\epsilon} + 2 \ln \frac{\bar{\mu}^2}{\omega^2} + 4 \right) - \frac{\omega^2 T^2}{48\pi} - \frac{\pi T^4}{30} + \mathcal{O} \left(\frac{T^6}{\omega^2} \right). \quad (\text{B.34})$$

B.10. $\rho_{\mathcal{I}_h}$

The sum-integral \mathcal{I}_h is defined as

$$\mathcal{I}_h \equiv \oint_{Q,R} \frac{P^4}{Q^2 R^2 [(Q - R)^2 + \lambda^2] (R - P)^2}. \quad (\text{B.35})$$

After the Matsubara sums the expression for the corresponding spectral function reads

$$\begin{aligned}
\rho_{\mathcal{I}_h}(\omega) &= \int_{\mathbf{q}, \mathbf{r}} \frac{\omega^4 \pi}{8qr E_{qr}} \left\{ \right. \\
&\quad \frac{1}{2r} \left[\delta(\omega - 2r) - \delta(\omega + 2r) \right] \times \\
&\quad \times \left[\left(\frac{1}{q + r - E_{qr}} + \frac{1}{q - r - E_{qr}} \right) (1 + 2n_r)(n_{qr} - n_q) \right. \\
&\quad \left. + \left(\frac{1}{q + r + E_{qr}} + \frac{1}{q - r + E_{qr}} \right) (1 + 2n_r)(1 + n_{qr} + n_q) \right] \\
&\quad - \left[\delta(\omega - q - r - E_{qr}) - \delta(\omega + q + r + E_{qr}) \right] \frac{(1 + n_{qr})(1 + n_q + n_r) + n_q n_r}{(q + r + E_{qr})(q - r + E_{qr})} \\
&\quad - \left[\delta(\omega - q - r + E_{qr}) - \delta(\omega + q + r - E_{qr}) \right] \frac{n_{qr}(1 + n_q + n_r) - n_q n_r}{(q + r - E_{qr})(q - r - E_{qr})} \\
&\quad - \left[\delta(\omega - q + r - E_{qr}) - \delta(\omega + q - r + E_{qr}) \right] \frac{n_r(1 + n_q + n_{qr}) - n_q n_{qr}}{(q - r + E_{qr})(q + r + E_{qr})} \\
&\quad \left. - \left[\delta(\omega + q - r - E_{qr}) - \delta(\omega - q + r + E_{qr}) \right] \frac{n_q(1 + n_r + n_{qr}) - n_r n_{qr}}{(q - r - E_{qr})(q + r - E_{qr})} \right\}. \quad (\text{B.36})
\end{aligned}$$

The main difference to the procedure described in appendix A is the handling of the “factorized powerlike” integrals, which are ultraviolet divergent, like in eq. (B.32). The expression reads

$$\rho_{\mathcal{I}_h}^{(\text{fz,p})}(\omega) \equiv \int_{\mathbf{q},\mathbf{r}} \frac{\omega^4 \pi}{16qr^2 E_{qr}} \delta(\omega - 2r) \left(\frac{1}{q+r+E_{qr}} + \frac{1}{-r+q+E_{qr}} \right) (1+2n_r). \quad (\text{B.37})$$

The \mathbf{r} -integral can be evaluated like in eq. (B.5), whereas the \mathbf{q} -integral can be re-expressed as a four-dimensional vacuum integral:

$$\int_{\mathbf{q}} \frac{1}{4qE_{qr}} \left(\frac{1}{\frac{\omega}{2}+q+E_{qr}} + \frac{1}{-\frac{\omega}{2}+q+E_{qr}} \right) = \int_Q \frac{1}{Q^2[(Q-R)^2+\lambda^2]} \Big|_{R=(\frac{\omega}{2}i, \frac{\omega}{2}\mathbf{e}_r)}. \quad (\text{B.38})$$

This can be verified by carrying out $\int \frac{dq_0}{2\pi}$ on the right-hand side. The value of the integral is familiar; in fact, since $O(D)$ rotational invariance implies that the result depends on λ^2 and R^2 only, and $R^2 = 0$ according to eq. (B.38), the simplest way to derive it is just by setting $R = 0$. In any case,

$$\int_Q \frac{1}{Q^2[(Q-R)^2+\lambda^2]} \Big|_{R=(\frac{\omega}{2}i, \frac{\omega}{2}\mathbf{e}_r)} = \frac{\mu^{-2\epsilon}}{(4\pi)^2} \left(\frac{1}{\epsilon} + \ln \frac{\bar{\mu}^2}{\lambda^2} + 1 + \mathcal{O}(\epsilon) \right), \quad (\text{B.39})$$

and in total (omitting $\mathcal{O}(\epsilon)$)

$$\rho_{\mathcal{I}_h}^{(\text{fz,p})}(\omega) = \frac{\omega^4 \mu^{-4\epsilon}}{4(4\pi)^3} (1+2n_{\frac{\omega}{2}}) \left(\frac{1}{\epsilon} + \ln \frac{\bar{\mu}^2}{\omega^2} + \ln \frac{\bar{\mu}^2}{\lambda^2} + 3 \right). \quad (\text{B.40})$$

The “factorized exponential” integrals can be worked out like before; the expression corresponding to eq. (A.14) becomes

$$\begin{aligned} \rho_{\mathcal{I}_h}^{(\text{fz,e})}(\omega) &= \frac{\omega^3}{2(4\pi)^3} (1+2n_{\frac{\omega}{2}}) \left\{ \int_0^\infty dq n_q \ln \left| \frac{2q\omega + \lambda^2}{2q\omega - \lambda^2} \right| \right. \\ &\quad \left. + \int_\lambda^\infty dq n_q \ln \left| \frac{q + \frac{\lambda^2}{\omega} - \sqrt{q^2 - \lambda^2}}{q + \frac{\lambda^2}{\omega} + \sqrt{q^2 - \lambda^2}} \right| \left| \frac{q - \frac{\lambda^2}{\omega} + \sqrt{q^2 - \lambda^2}}{q - \frac{\lambda^2}{\omega} - \sqrt{q^2 - \lambda^2}} \right| \right\}. \end{aligned} \quad (\text{B.41})$$

For the phase space integrals the expression corresponding to eq. (A.30) reads

$$\begin{aligned} \rho_{\mathcal{I}_h}^{(\text{ps})}(\omega) &= \frac{\omega^3}{2(4\pi)^3} (e^\omega - 1) \left\{ \right. \\ &\quad \text{(i)} \quad - \int_{\frac{\lambda^2}{2\omega}}^{\frac{\omega}{2}} dq \int_{\frac{\lambda^2}{4q}}^{\frac{\omega(\omega-2q)+\lambda^2}{2\omega}} dr \mathbb{P}\left(\frac{1}{r}\right) n_{\frac{\omega}{2}-q} n_{\frac{\omega}{2}-r} n_{q+r} \\ &\quad \text{(ii)} \quad + \int_0^\infty dq \int_{\frac{\lambda^2}{4q}}^\infty dr \mathbb{P}\left(\frac{1}{r}\right) n_{\frac{\omega}{2}+q} n_{\frac{\omega}{2}+r} n_{q+r} e^{q+r} \\ &\quad \text{(iii)} \quad + \int_{\frac{\omega}{2}}^\infty dq \int_{-\frac{\lambda^2}{4q}}^{\frac{\omega(2q-\omega)-\lambda^2}{2\omega}} dr \mathbb{P}\left(\frac{1}{r} - \frac{1}{q}\right) n_{q-\frac{\omega}{2}} n_{\frac{\omega}{2}+r} n_{q-r} e^{q-\frac{\omega}{2}} \left. \right\}. \end{aligned} \quad (\text{B.42})$$

Subsequently we can reflect the range $q < r$ to $q > r$ in cases (i) and (ii). The subtractions of the potentially divergent parts are carried out like in eqs. (A.35), (A.40), (A.44); the integrals over y and r are a bit more complicated than with \mathcal{I}_j but can be carried out.

Once the pieces are collected together, we can again identify a “vacuum part”, a “1-dimensional integral”, and a “2-dimensional integral”. The contribution to the vacuum part from the phase space integrals reads

$$\frac{\omega^3}{2(4\pi)^3} (1 + 2n_{\frac{\omega}{2}}) \int_{\lambda}^{\frac{\omega}{2}} dq \ln \left| \frac{q - \sqrt{q^2 - \lambda^2}}{q + \sqrt{q^2 - \lambda^2}} \right| \approx \frac{\omega^4}{4(4\pi)^3} (1 + 2n_{\frac{\omega}{2}}) \left(\ln \frac{\lambda^2}{\omega^2} + 2 \right). \quad (\text{B.43})$$

Summing together with eq. (B.40), we obtain the first row of eq. (B.44). The most tedious work concerns the handling of the 1-dimensional integrals; however in principle everything proceeds like in sec. A.5, and eventually the λ 's cancel or can safely be put to zero. Altogether we obtain

$$\begin{aligned} \frac{2(4\pi)^3 \rho_{\mathcal{I}_h}(\omega)}{\omega^3(1 + 2n_{\frac{\omega}{2}})} &= \frac{\omega \mu^{-4\epsilon}}{2} \left(\frac{1}{\epsilon} + 2 \ln \frac{\bar{\mu}^2}{\omega^2} + 5 \right) \\ &+ \int_0^{\frac{\omega}{4}} dq n_q \left\{ -2 \ln \left(1 - \frac{2q}{\omega} \right) + 2 \ln \left(1 + \frac{2q}{\omega} \right) + \frac{\omega}{2} \left(\frac{1}{q + \frac{\omega}{2}} + \frac{1}{q - \frac{\omega}{2}} \right) \right\} \\ &+ \int_{\frac{\omega}{4}}^{\frac{\omega}{2}} dq n_q \left\{ -3 \ln \left(1 - \frac{2q}{\omega} \right) + 2 \ln \left(1 + \frac{2q}{\omega} \right) + \ln \left(\frac{2q}{\omega} \right) + \frac{\omega}{2} \frac{1}{q + \frac{\omega}{2}} - 2 \right\} \\ &+ \int_{\frac{\omega}{2}}^{\infty} dq n_q \left\{ -4 \ln \left(\frac{2q}{\omega} - 1 \right) + 2 \ln \left(1 + \frac{2q}{\omega} \right) + 2 \ln \left(\frac{2q}{\omega} \right) + \frac{\omega}{2} \left(\frac{1}{q + \frac{\omega}{2}} - \frac{1}{q} \right) - 1 \right\} \\ &+ \int_0^{\frac{\omega}{2}} dq \int_0^{\frac{\omega}{4} - |q - \frac{\omega}{4}|} dr \left(-\frac{1}{q} - \frac{1}{r} \right) \frac{n_{\frac{\omega}{2} - q} n_{q+r} (1 + n_{\frac{\omega}{2} - r})}{n_r^2} \\ &+ \int_{\frac{\omega}{2}}^{\infty} dq \int_0^{q - \frac{\omega}{2}} dr \left(\frac{1}{q} - \frac{1}{r} \right) \frac{n_{q - \frac{\omega}{2}} (1 + n_{q-r}) (n_q - n_{r + \frac{\omega}{2}})}{n_r n_{-\frac{\omega}{2}}} \\ &+ \int_0^{\infty} dq \int_0^q dr \left(\frac{1}{q} + \frac{1}{r} \right) \frac{(1 + n_{q + \frac{\omega}{2}}) n_{q+r} n_{r + \frac{\omega}{2}}}{n_r^2} + \mathcal{O}(\lambda \ln \lambda). \end{aligned} \quad (\text{B.44})$$

For $\omega \gg \pi T$ it can be checked numerically that this goes over to the known ultraviolet asymptotics [36]

$$\rho_{\mathcal{I}_h}^{\text{OPE}}(\omega) = \frac{\omega^4 \mu^{-4\epsilon}}{4(4\pi)^3} \left(\frac{1}{\epsilon} + 2 \ln \frac{\bar{\mu}^2}{\omega^2} + 5 \right) + \frac{\omega^2 T^2}{192\pi} - \frac{\pi T^4}{360} + \mathcal{O}\left(\frac{T^6}{\omega^2}\right). \quad (\text{B.45})$$

B.11. $\rho_{\mathcal{I}_i}$

The sum-integral \mathcal{I}_i , is defined as

$$\mathcal{I}_i \equiv \oint_{Q,R} \frac{4(Q \cdot P)^2}{Q^2 R^2 [(Q - R)^2 + \lambda^2] (R - P)^2}. \quad (\text{B.46})$$

At zero spatial momentum, $4(Q \cdot P)^2 = 4q_n^2 p_n^2 = 4(Q^2 - q^2)p_n^2$, so that

$$\mathcal{I}_i = \oint_{Q,R} \frac{4p_n^2}{R^2[(Q-R)^2 + \lambda^2](R-P)^2} - \oint_{Q,R} \frac{4p_n^2 q^2}{Q^2 R^2[(Q-R)^2 + \lambda^2](R-P)^2}. \quad (\text{B.47})$$

The Matsubara sums here are equivalent to those in \mathcal{I}_b and \mathcal{I}_h , and we obtain

$$\begin{aligned} \rho_{\mathcal{I}_i}(\omega) = \int_{\mathbf{q}, \mathbf{r}} \frac{\omega^2 q \pi}{2r E_{qr}} \Big\{ & \frac{1}{2r} \left[\delta(\omega - 2r) - \delta(\omega + 2r) \right] \times \\ & \times \left[\left(-\frac{2}{q} \right) (1 + 2n_r)(1 + 2n_{qr}) \right. \\ & + \left(\frac{1}{q + r - E_{qr}} + \frac{1}{q - r - E_{qr}} \right) (1 + 2n_r)(n_{qr} - n_q) \\ & + \left. \left(\frac{1}{q + r + E_{qr}} + \frac{1}{q - r + E_{qr}} \right) (1 + 2n_r)(1 + n_{qr} + n_q) \right] \\ & - \left[\delta(\omega - q - r - E_{qr}) - \delta(\omega + q + r + E_{qr}) \right] \frac{(1 + n_{qr})(1 + n_q + n_r) + n_q n_r}{(q + r + E_{qr})(q - r + E_{qr})} \\ & - \left[\delta(\omega - q - r + E_{qr}) - \delta(\omega + q + r - E_{qr}) \right] \frac{n_{qr}(1 + n_q + n_r) - n_q n_r}{(q + r - E_{qr})(q - r - E_{qr})} \\ & - \left[\delta(\omega - q + r - E_{qr}) - \delta(\omega + q - r + E_{qr}) \right] \frac{n_r(1 + n_q + n_{qr}) - n_q n_{qr}}{(q - r + E_{qr})(q + r + E_{qr})} \\ & - \left. \left[\delta(\omega + q - r - E_{qr}) - \delta(\omega - q + r + E_{qr}) \right] \frac{n_q(1 + n_r + n_{qr}) - n_r n_{qr}}{(q - r - E_{qr})(q + r - E_{qr})} \right\}. \quad (\text{B.48}) \end{aligned}$$

Like with $\rho_{\mathcal{I}_h}$ the “factorized powerlike” integrals are ultraviolet divergent; the expression reads

$$\rho_{\mathcal{I}_i}^{(\text{fz,p})}(\omega) \equiv \int_{\mathbf{q}, \mathbf{r}} \frac{\omega^2 q \pi}{4r^2 E_{qr}} \delta(\omega - 2r) \left(-\frac{2}{q} + \frac{1}{q + r + E_{qr}} + \frac{1}{-r + q + E_{qr}} \right) (1 + 2n_r). \quad (\text{B.49})$$

The \mathbf{r} -integral is the same as in eq. (B.5), whereas the non-trivial \mathbf{q} -integral can be re-expressed as a four-dimensional vacuum integral:

$$\int_{\mathbf{q}} \frac{q}{4E_{qr}} \left(\frac{1}{\frac{\omega}{2} + q + E_{qr}} + \frac{1}{-\frac{\omega}{2} + q + E_{qr}} \right) = \int_Q \frac{q^2}{Q^2[(Q-R)^2 + \lambda^2]} \Big|_{R=(\frac{\omega}{2}i, \frac{\omega}{2}\mathbf{e}_r)}. \quad (\text{B.50})$$

This can be verified by carrying out $\int \frac{dq_0}{2\pi}$ on the right-hand side of eq. (B.50). Once expressed this way, we can make use of $O(D)$ rotational covariance, in order to reduce the “tensor” structure in the numerator to scalar integrals. The result is then a function of λ^2 and R^2 only; given that $R^2 = 0$, we need the leading term in a Taylor-expansion around this point. A few steps lead to

$$\int_Q \frac{q^2}{Q^2[(Q-R)^2 + \lambda^2]} \Big|_{R=(\frac{\omega}{2}i, \frac{\omega}{2}\mathbf{e}_r)} = \frac{\omega^2 \mu^{-2\epsilon}}{12(4\pi)^2} \left(\frac{1}{\epsilon} + \ln \frac{\bar{\mu}^2}{\lambda^2} + \frac{1}{3} + \mathcal{O}(\epsilon) \right), \quad (\text{B.51})$$

and in total

$$\rho_{\mathcal{I}_1}^{(\text{fz,p})}(\omega) = \frac{\omega^4 \mu^{-4\epsilon}}{12(4\pi)^3} (1 + 2n_{\frac{\omega}{2}}) \left(\frac{1}{\epsilon} + \ln \frac{\bar{\mu}^2}{\omega^2} + \ln \frac{\bar{\mu}^2}{\lambda^2} + \frac{7}{3} + \mathcal{O}(\epsilon) \right). \quad (\text{B.52})$$

The “factorized exponential” integrals can be worked out like before; the expression corresponding to eq. (A.14) becomes

$$\begin{aligned} \rho_{\mathcal{I}_1}^{(\text{fz,e})}(\omega) &= \frac{2\omega}{(4\pi)^3} (1 + 2n_{\frac{\omega}{2}}) \left\{ \int_0^\infty dq n_q q^2 \ln \left| \frac{2q\omega + \lambda^2}{2q\omega - \lambda^2} \right| + \int_\lambda^\infty dq n_q \times \right. \\ &\quad \times \left[\left(q + \frac{\omega}{2} \right)^2 \ln \left| \frac{q + \frac{\lambda^2}{\omega} - \sqrt{q^2 - \lambda^2}}{q + \frac{\lambda^2}{\omega} + \sqrt{q^2 - \lambda^2}} \right| + \left(q - \frac{\omega}{2} \right)^2 \ln \left| \frac{q - \frac{\lambda^2}{\omega} + \sqrt{q^2 - \lambda^2}}{q - \frac{\lambda^2}{\omega} - \sqrt{q^2 - \lambda^2}} \right| \right] \Big\}. \end{aligned} \quad (\text{B.53})$$

For the phase space integrals the expression corresponding to eq. (A.30) reads

$$\begin{aligned} \rho_{\mathcal{I}_1}^{(\text{ps})}(\omega) &= \frac{2\omega}{(4\pi)^3} (e^\omega - 1) \left\{ \right. \\ &\quad \text{(i)} \quad - \int_{\frac{\lambda^2}{2\omega}}^{\frac{\omega}{2}} dq \int_{\frac{\lambda^2}{4q}}^{\frac{\omega(\omega-2q)+\lambda^2}{2\omega}} dr \mathbb{P} \left[\frac{(q - \frac{\omega}{2})^2}{r} \right] n_{\frac{\omega}{2}-q} n_{\frac{\omega}{2}-r} n_{q+r} \\ &\quad \text{(ii)} \quad + \int_0^\infty dq \int_{\frac{\lambda^2}{4q}}^\infty dr \mathbb{P} \left[\frac{(q + \frac{\omega}{2})^2}{r} \right] n_{\frac{\omega}{2}+q} n_{\frac{\omega}{2}+r} n_{q+r} e^{q+r} \\ &\quad \text{(iii)} \quad + \int_{\frac{\omega}{2}}^\infty dq \int_{-\frac{\lambda^2}{4q}}^{\frac{\omega(2q-\omega)-\lambda^2}{2\omega}} dr \mathbb{P} \left[\frac{(q - \frac{\omega}{2})^2}{r} - \frac{(r + \frac{\omega}{2})^2}{q} \right] n_{q-\frac{\omega}{2}} n_{\frac{\omega}{2}+r} n_{q-r} e^{q-\frac{\omega}{2}} \Big\}. \end{aligned} \quad (\text{B.54})$$

Subsequently we can reflect the range $q < r$ to $q > r$ in cases (i) and (ii). The subtractions of the potentially divergent parts are carried out like in eqs. (A.35), (A.40), (A.44); the integrals over y and r are a bit more complicated than with \mathcal{I}_j but can be carried out.

Once the pieces are collected together, we can again identify a “vacuum part”, a “1-dimensional integral”, and a “2-dimensional integral”. The contribution to the vacuum part from the phase space integrals reads

$$\begin{aligned} &\frac{2\omega}{(4\pi)^3} (1 + 2n_{\frac{\omega}{2}}) \int_\lambda^{\frac{\omega}{2}} dq \left[\left(q - \frac{\omega}{2} \right)^2 \ln \left| \frac{q - \sqrt{q^2 - \lambda^2}}{q + \sqrt{q^2 - \lambda^2}} \right| - \omega q + \frac{3}{2} q^2 \right] \\ &\approx \frac{\omega^4}{12(4\pi)^3} (1 + 2n_{\frac{\omega}{2}}) \left(\ln \frac{\lambda^2}{\omega^2} + \frac{13}{6} \right). \end{aligned} \quad (\text{B.55})$$

Summing together with eq. (B.52), we obtain the first row of eq. (B.56). The most tedious work concerns the handling of the 1-dimensional integrals; however in principle everything

proceeds like in sec. A.5, and eventually the λ 's cancel or can safely be put to zero. Altogether we obtain

$$\begin{aligned}
\frac{(4\pi)^3 \rho_{\mathcal{I}_1}(\omega)}{2\omega(1+2n_{\frac{\omega}{2}})} &= \frac{\omega^3 \mu^{-4\epsilon}}{24} \left(\frac{1}{\epsilon} + 2 \ln \frac{\bar{\mu}^2}{\omega^2} + \frac{9}{2} \right) \\
&+ \int_0^{\frac{\omega}{4}} dq n_q \left\{ - \left[q^2 + \left(q - \frac{\omega}{2} \right)^2 \right] \ln \left(1 - \frac{2q}{\omega} \right) + \left[q^2 + \left(q + \frac{\omega}{2} \right)^2 \right] \ln \left(1 + \frac{2q}{\omega} \right) \right. \\
&\quad \left. + \frac{\omega^3}{24} \left(\frac{1}{q + \frac{\omega}{2}} + \frac{1}{q - \frac{\omega}{2}} \right) - \frac{8\omega q}{3} \right\} \\
&+ \int_{\frac{\omega}{4}}^{\frac{\omega}{2}} dq n_q \left\{ - \left[2q^2 + \left(q - \frac{\omega}{2} \right)^2 \right] \ln \left(1 - \frac{2q}{\omega} \right) + \left[q^2 + \left(q + \frac{\omega}{2} \right)^2 \right] \ln \left(1 + \frac{2q}{\omega} \right) \right. \\
&\quad \left. + q^2 \ln \left(\frac{2q}{\omega} \right) + \frac{\omega^3}{24} \left(\frac{1}{q + \frac{\omega}{2}} \right) - \frac{\omega^2}{6} - \frac{5\omega q}{2} - \frac{2q^2}{3} \right\} \\
&+ \int_{\frac{\omega}{2}}^{\infty} dq n_q \left\{ -2 \left[q^2 + \left(q - \frac{\omega}{2} \right)^2 \right] \ln \left(\frac{2q}{\omega} - 1 \right) + \left[q^2 + \left(q + \frac{\omega}{2} \right)^2 \right] \ln \left(1 + \frac{2q}{\omega} \right) \right. \\
&\quad \left. + \left[q^2 + \left(q - \frac{\omega}{2} \right)^2 \right] \ln \left(\frac{2q}{\omega} \right) + \frac{\omega^3}{24} \left(\frac{1}{q + \frac{\omega}{2}} - \frac{1}{q} \right) - \frac{\omega^2}{6} - \frac{3\omega q}{2} - \frac{11q^2}{6} \right\} \\
&+ \int_0^{\frac{\omega}{2}} dq \int_0^{\frac{\omega}{4} - |q - \frac{\omega}{4}|} dr \left[- \frac{(r - \frac{\omega}{2})^2}{q} - \frac{(q - \frac{\omega}{2})^2}{r} \right] \frac{n_{\frac{\omega}{2}-q} n_{q+r} (1 + n_{\frac{\omega}{2}-r})}{n_r^2} \\
&+ \int_{\frac{\omega}{2}}^{\infty} dq \int_0^{q - \frac{\omega}{2}} dr \left[\frac{(r + \frac{\omega}{2})^2}{q} - \frac{(q - \frac{\omega}{2})^2}{r} \right] \frac{n_{q-\frac{\omega}{2}} (1 + n_{q-r}) (n_q - n_{r+\frac{\omega}{2}})}{n_r n_{-\frac{\omega}{2}}} \\
&+ \int_0^{\infty} dq \int_0^q dr \left[\frac{(r + \frac{\omega}{2})^2}{q} + \frac{(q + \frac{\omega}{2})^2}{r} \right] \frac{(1 + n_{q+\frac{\omega}{2}}) n_{q+r} n_{r+\frac{\omega}{2}}}{n_r^2} + \mathcal{O}(\lambda \ln \lambda) . \quad (\text{B.56})
\end{aligned}$$

Numerically it can be checked that for $\omega \gg \pi T$ this approaches the known asymptotics [36]

$$\rho_{\mathcal{I}_1}^{\text{OPE}}(\omega) = \frac{\omega^4 \mu^{-4\epsilon}}{12(4\pi)^3} \left(\frac{1}{\epsilon} + 2 \ln \frac{\bar{\mu}^2}{\omega^2} + \frac{9}{2} \right) - \frac{\omega^2 T^2}{96\pi} + \frac{\pi T^4}{120} + \mathcal{O}\left(\frac{T^6}{\omega^2}\right) . \quad (\text{B.57})$$

C. Hard Thermal Loop resummation

We give here some details for the Hard Thermal Loop (HTL) resummation discussed in sec. 5.2, which is important when $\omega \sim gT$ and could allow us to postpone the breakdown of the perturbative series down to lower frequencies ($\omega \lesssim g^2T/\pi$).

In general, HTL resummation is needed both for propagators and vertices; for instance, the coupling of a photon to quarks and gluons gets modified by a correction $\propto g^2T^2$ and needs to be taken into account at leading order if the photons are soft [43]. On the other hand, the coupling of a heavy quark to gluons does *not* get modified even at NLO within HTL perturbation theory [48]. For us relevant is the coupling of a dilaton or an axion to gluons. For “hard” axions it can trivially be argued that no vertex modification is needed [55] but here we are concerned with “soft” dilatons or axions ($\omega \sim gT$, $\mathbf{p} = \mathbf{0}$). It is nevertheless our belief that no vertex modification is needed, neither for dilatons nor for axions. Concrete indications in this direction are that even without a vertex correction we find results which exactly cancel the leading infrared divergences from the QCD results in both channels (cf. sec. 5.2), and that the results satisfy the expected sum rules (cf. last paragraph of sec. 5.4). In addition, we have checked the absence of vertex corrections through explicit 1-loop computations.³ We assume that simple theoretical arguments could also be given for the absence of vertex corrections but at the time of writing these evade us. No further details are provided here because, strictly speaking, it would not even be necessary for us to treat the infrared regime correctly.

C.1. Contractions

With the logic outlined above, the only ingredient needed from the HTL theory is the gauge field propagator, which takes the form

$$\langle A_\mu^a(X) A_\nu^b(Y) \rangle = \delta^{ab} \oint_Q e^{iQ \cdot (X-Y)} \left[\frac{\mathbb{P}_{\mu\nu}^T(Q)}{Q^2 + \Pi_T(Q)} + \frac{\mathbb{P}_{\mu\nu}^E(Q)}{Q^2 + \Pi_E(Q)} + \frac{\xi Q_\mu Q_\nu}{Q^4} \right], \quad (\text{C.1})$$

where ξ is a gauge parameter. The two independent projection operators are defined by

$$\mathbb{P}_{\mu\nu}^T(Q) \equiv \delta_{\mu i} \delta_{\nu j} \left(\delta_{ij} - \frac{q_i q_j}{q^2} \right), \quad \mathbb{P}_{\mu\nu}^E(Q) \equiv \delta_{\mu\nu} - \frac{Q_\mu Q_\nu}{Q^2} - \mathbb{P}_{\mu\nu}^T(Q), \quad (\text{C.2})$$

whereas the self-energies read

$$\Pi_T(Q) = \frac{m_E^2}{D-2} \int_z \left(1 - \frac{Q^2}{q^2} \frac{qz}{iq_n + qz} \right), \quad \Pi_E(Q) = \frac{m_E^2 Q^2}{q^2} \int_z \frac{qz}{iq_n + qz}, \quad (\text{C.3})$$

³According to ref. [56], considering a three-leg process in Feynman gauge Hard Thermal Loops could arise from structures of the types $\mathcal{F}_K \frac{K_\mu K_\nu K_\rho}{K^2(K-P)^2(K-Q)^2}$, $\mathcal{F}_K \frac{K_\mu K_\nu}{K^2(K-P)^2}$, as well as $\mathcal{F}_K \frac{1}{K^2}$. In the dilaton case individual graphs do give contributions of the latter two types but these cancel in the sum; in the axion case we find no such contributions.

with the Debye mass parameter given by

$$m_E^2 \equiv g^2 N_c (D-2)^2 \int_r \frac{n_B(r)}{r} . \quad (\text{C.4})$$

For $D = 4$ this reproduces the expression given in eq. (5.13).⁴

In the main body of the text, a *next-to-leading order* computation was described. For $\omega \sim gT$, however, the series representation breaks down. Through the HTL method we then resum infinite orders to a single contribution, which is of the *leading order* within the HTL power counting. At this order, all the integrals met are ultraviolet finite, so that we could set $D = 4$; for generality, however, we mostly show D in its original form.

When the propagator takes the form in eq. (C.1), the contractions leading to eqs. (3.2), (3.3) need to be carried out anew. Because of the non-trivial structure of eq. (C.1), this is somewhat tedious even at leading order. The problem gets simplified if we set $\mathbf{p} \rightarrow \mathbf{0}$ from the outset; then

$$P_\mu \mathbb{P}_{\mu\nu}^T(Q) = Q_\mu \mathbb{P}_{\mu\nu}^T(Q) = 0 . \quad (\text{C.6})$$

Also, a combination appearing frequently, particularly in the χ -channel, reduces to $Q^2 P^2 - (Q \cdot P)^2 = q^2 p_n^2$. Thereby we obtain (making also use of $[Q \cdot (Q - P)]^2 = Q^2(Q - P)^2 - [Q^2 P^2 - (Q \cdot P)^2]$ in order to identify a common part in both channels)

$$\begin{aligned} \frac{\tilde{G}_\theta^{\text{HTL}}(P)}{4d_A c_\theta^2} &= -2(D-2)g^4 p_n^2 \oint_Q \frac{q^2}{\Delta_T(Q)\Delta_T(Q-P)} \\ &\quad + 2g^4 \oint_Q \left\{ \frac{Q^2(Q-P)^2}{\Delta_E(Q)\Delta_E(Q-P)} + \frac{(D-2)Q^2(Q-P)^2}{\Delta_T(Q)\Delta_T(Q-P)} \right\} , \end{aligned} \quad (\text{C.7})$$

$$\frac{\tilde{G}_\chi^{\text{HTL}}(P)}{-16d_A c_\chi^2 (D-3)} = -2(D-2)g^4 p_n^2 \oint_Q \frac{q^2}{\Delta_T(Q)\Delta_T(Q-P)} , \quad (\text{C.8})$$

where we have defined $\Delta_T(Q) \equiv Q^2 + \Pi_T(Q)$ and $\Delta_E(Q) \equiv Q^2 + \Pi_E(Q)$.

C.2. Naive HTL computation

As discussed around eq. (5.11), we need both a “naive” and a “resummed” version of the HTL result. We start with the naive computation; this means that m_E^2 is treated as a small quantity, of $\mathcal{O}(g^2)$, and the HTL-result is “re-expanded” in g^2 up to NLO.

⁴We have employed a notation valid in an arbitrary dimension, with the integration measures defined as ($d \equiv D - 1 = 3 - 2\epsilon$)

$$\int_r \equiv \frac{2}{(4\pi)^{\frac{d}{2}} \Gamma(\frac{d}{2})} \int_0^\infty dr r^{d-1} , \quad \int_z \equiv \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{d-1}{2})} \int_{-1}^{+1} dz (1-z^2)^{\frac{d-3}{2}} , \quad (\text{C.5})$$

such that $\int_r = \int_r \int_z$ and $\int_z 1 = 1$.

Now, since the HTL theory (correctly) modifies physics only at soft scales, one should be careful not to apply HTL resummation to parts of the phase space where it is not needed. In real-time computations, it is notoriously difficult to identify the relevant parts in advance. Therefore, in the following we carry out the HTL resummation on all scales, whereby the HTL results show dependence on *two* scales, both the “soft” m_E as well as the “hard” πT . The unwanted dependence on πT is easily dropped *a posteriori*, cf. text below eq. (C.25).

With this philosophy, we start with the pseudoscalar (χ) channel, which is simpler (cf. eq. (C.8)). It will be checked afterwards that the additional terms in eq. (C.7) give no contribution in the “naive” limit (apart from “contact terms” of the type discussed in sec. 5.4).

First of all, at leading order, $\Delta_T(Q) = Q^2$. Then the structure in eq. (C.8) is almost the same as the master sum-integral \mathcal{J}_b of eq. (B.3), and we get

$$\begin{aligned} \text{Im} \left\{ \oint_Q \frac{-p_n^2 q^2}{Q^2(Q-P)^2} \right\}_{P \rightarrow (-i[\omega+i0^+], \mathbf{0})} &= \int_{\mathbf{q}} \frac{\omega^2 \pi}{4} [\delta(\omega - 2q) - \delta(\omega + 2q)] (1 + 2n_q) \\ &= \frac{\omega^4}{64\pi} (1 + 2n_{\frac{\omega}{2}}) + \mathcal{O}(\epsilon) . \end{aligned} \quad (\text{C.9})$$

Adding the factor $2(D-2)g^4$ and setting $D \rightarrow 4$ we get

$$\frac{-\rho_\chi^{\text{HTL}}(\omega)}{16d_A c_\chi^2} = \frac{\pi g^4 \omega^4}{(4\pi)^2} (1 + 2n_{\frac{\omega}{2}}) + \mathcal{O}(g^6) , \quad (\text{C.10})$$

which indeed agrees with the leading-order term in eq. (4.2).

At NLO, we can expand Δ_T to first order in Π_T . Substituting subsequently $Q \rightarrow P - Q$ yields

$$\left. \frac{\tilde{G}_\chi^{\text{HTL}}(P)}{-16d_A c_\chi^2 (D-3)} \right|_{\mathcal{O}(g^6)} = 4(D-2)g^4 p_n^2 \oint_Q \frac{q^2 \Pi_T(Q)}{Q^4(Q-P)^2} , \quad (\text{C.11})$$

where Π_T is to be inserted from eq. (C.3). There are two structures; for the simpler one we proceed in analogy with eqs. (B.17), (B.18):

$$\begin{aligned} \oint_Q \frac{q^2}{Q^4(Q-P)^2} &\rightarrow -\frac{1}{2} \frac{d}{d\lambda^2} \oint_Q \frac{q^2}{[Q^2 + \lambda^2][(Q-P)^2 + \lambda^2]} \\ &\rightarrow -\frac{1}{2} \frac{d}{d\lambda^2} \int_{\mathbf{q}} \frac{\pi q^2}{4E_q^2} \delta(\omega - 2E_q) (1 + 2n_{E_q}) \\ &= -\frac{1}{2} \frac{d}{d\lambda^2} \left\{ \frac{(\omega^2 - 4\lambda^2)^{\frac{3}{2}}}{64\pi\omega} (1 + 2n_{\frac{\omega}{2}}) \theta(\omega - 2\lambda) + \mathcal{O}(\epsilon) \right\} \\ &\rightarrow \frac{3\pi}{4(4\pi)^2} (1 + 2n_{\frac{\omega}{2}}) . \end{aligned} \quad (\text{C.12})$$

The second term is a bit more complicated, and it is useful to introduce an infrared regulator:

$$\begin{aligned}
& -\oint_Q \int_z \frac{1}{Q^2(Q-P)^2} \frac{qz}{iq_n + qz} \rightarrow -\int_z \oint_Q \frac{1}{(q_n - p_n)^2 + q^2} \frac{1}{q_n^2 + E_q^2} \frac{qz}{iq_n + qz} \\
& = -\int_{z,q} T \sum_{q_n} T \sum_{r_n} \frac{\beta \delta_{r_n + p_n - q_n}}{(r_n^2 + q^2)(q_n^2 + E_q^2)} \frac{1}{2} \left(\frac{qz}{iq_n + qz} + \frac{qz}{-iq_n + qz} \right) \\
& = -\int_{z,q} \int_0^\beta d\tau \frac{q^2 z^2 e^{ip_n \tau}}{E_q^2 - q^2 z^2} T \sum_{r_n} \frac{e^{ir_n \tau}}{r_n^2 + q^2} T \sum_{q_n} \left(\frac{e^{-iq_n \tau}}{q_n^2 + q^2 z^2} - \frac{e^{-iq_n \tau}}{q_n^2 + E_q^2} \right), \quad (\text{C.13})
\end{aligned}$$

where $E_q^2 \equiv q^2 + \lambda^2$; we symmetrized in $z \leftrightarrow -z$; represented the Kronecker- δ as an integral; and partial-fractioned the dependence on q_n^2 . The sums can be carried out, e.g.

$$T \sum_{r_n} \frac{e^{ir_n \tau}}{r_n^2 + q^2} = \frac{n_q}{2q} \left[e^{q\tau} + e^{q(\beta-\tau)} \right], \quad (\text{C.14})$$

and subsequently also the integral over τ , yielding

$$\begin{aligned}
I(p_n) & \equiv -\int_{z,q} \frac{q^2 z^2}{\lambda^2 + q^2(1-z^2)} \frac{n_q}{4q} \times \\
& \times \left\{ \frac{n_{qz}}{qz} \left[\left(e^{q(1+z)\beta} - 1 \right) \left(\frac{1}{-ip_n + q(1+z)} + \frac{1}{ip_n + q(1+z)} \right) \right. \right. \\
& \quad \left. \left. + \left(e^{q\beta} - e^{qz\beta} \right) \left(\frac{1}{-ip_n + q(1-z)} + \frac{1}{ip_n + q(1-z)} \right) \right] \right. \\
& \quad \left. - \frac{n_{E_q}}{E_q} \left[\left(e^{(q+E_q)\beta} - 1 \right) \left(\frac{1}{-ip_n + q + E_q} + \frac{1}{ip_n + q + E_q} \right) \right. \right. \\
& \quad \left. \left. + \left(e^{q\beta} - e^{E_q\beta} \right) \left(\frac{1}{-ip_n + q - E_q} + \frac{1}{ip_n + q - E_q} \right) \right] \right\}. \quad (\text{C.15})
\end{aligned}$$

We can now set $p_n \rightarrow -i[\omega + i0^+]$ and take the imaginary part; for $\omega > \lambda > 0$, only three channels contribute. If we also substitute $z \rightarrow -z$ in one of them, and make use of the δ -constraints in order to re-express the arguments of the Bose distributions, the expression reduces to

$$\begin{aligned}
\text{Im } I(-i[\omega + i0^+]) & = -\frac{\pi}{4} \int_{z,q} \frac{z^2}{\lambda^2 + q^2(1-z^2)} (1 + n_q + n_{\omega-q}) \\
& \times \left[\frac{2}{z} \delta(\omega - q - qz) - \frac{q}{E_q} \delta(\omega - q - E_q) \right]. \quad (\text{C.16})
\end{aligned}$$

The integrals over z can be carried out:

$$\frac{1}{2} \int_{-1}^{+1} dz \frac{2z \delta(\omega - q - qz)}{\lambda^2 + q^2(1-z^2)} = \frac{(\omega - q) \theta(q - \frac{\omega}{2})}{q^2(\lambda^2 + 2\omega q - \omega^2)}, \quad (\text{C.17})$$

$$\frac{1}{2} \int_{-1}^{+1} dz \frac{z^2}{\lambda^2 + q^2(1-z^2)} = -\frac{1}{q^2} \left(1 + \frac{E_q}{2q} \ln \frac{E_q - q}{E_q + q} \right). \quad (\text{C.18})$$

The q -integral over $\delta(\omega - q - E_q)$ yields

$$\int_0^\infty dq \phi(q, E_q) \delta(\omega - q - E_q) = \frac{E_q}{q + E_q} \phi(q, E_q) \Big|_{q=\frac{\omega^2-\lambda^2}{2\omega}, E_q=\frac{\omega^2+\lambda^2}{2\omega}} , \quad (\text{C.19})$$

whereas the other q -integral is split into the ranges $(\frac{\omega}{2}, \omega)$ and (ω, ∞) . In the former range, we add and subtract $1 + 2n_{\frac{\omega}{2}}$ from the Bose distributions, whereas in the latter range, we write $1 + n_{\omega-q} = -n_{q-\omega}$ in order to have a positive argument. In total, setting $\lambda \rightarrow 0$ whenever possible,

$$\begin{aligned} \text{Im } I(-i[\omega + i0^+]) \approx & -\frac{1}{8\pi} \left\{ \int_{\frac{\omega}{2}}^\omega dq \frac{\omega - q}{2\omega q - \omega^2} (n_q + n_{\omega-q} - 2n_{\frac{\omega}{2}}) \right. \\ & + \int_\omega^\infty dq \frac{\omega - q}{2\omega q - \omega^2} (n_q - n_{q-\omega}) \\ & + \int_{\frac{\omega}{2}}^\omega dq \frac{\omega - q}{\lambda^2 + 2\omega q - \omega^2} (1 + 2n_{\frac{\omega}{2}}) \\ & \left. + \frac{E_q}{q + E_q} \left(\frac{q}{E_q} + \frac{1}{2} \ln \frac{E_q - q}{E_q + q} \right) (1 + n_q + n_{E_q}) \Big|_{q=\frac{\omega^2-\lambda^2}{2\omega}} \right\} . \end{aligned} \quad (\text{C.20})$$

The last two terms contain a logarithmic divergence, but their sum is finite:

$$\int_{\frac{\omega}{2}}^\omega dq \frac{\omega - q}{\lambda^2 + 2\omega q - \omega^2} = \frac{1}{4} \ln \frac{\omega^2}{\lambda^2} - \frac{1}{4} + \mathcal{O}(\lambda \ln \lambda) , \quad (\text{C.21})$$

$$\frac{E_q}{q + E_q} \left(\frac{q}{E_q} + \frac{1}{2} \ln \frac{E_q - q}{E_q + q} \right) \Big|_{q=\frac{\omega^2-\lambda^2}{2\omega}} = \frac{1}{4} \ln \frac{\lambda^2}{\omega^2} + \frac{1}{2} + \mathcal{O}(\lambda \ln \lambda) . \quad (\text{C.22})$$

If we also write

$$n_q + n_{\omega-q} - 2n_{\frac{\omega}{2}} = (1 + 2n_{\frac{\omega}{2}}) \frac{n_q(1 + n_{\omega-q})}{n_{q-\frac{\omega}{2}}^2} , \quad (\text{C.23})$$

$$n_q - n_{q-\omega} = -(1 + 2n_{\frac{\omega}{2}}) \frac{n_q(1 + n_{q-\omega})}{n_{\frac{\omega}{2}}^2} , \quad (\text{C.24})$$

and combine then with the contribution in eq. (C.12) as well as the leading-order term from eq. (C.10), the full result becomes

$$\begin{aligned} \frac{-\rho_\chi^{\text{HTL}}(\omega)}{16d_A c_\chi^2} = & \frac{\pi g^4}{(4\pi)^2} (1 + 2n_{\frac{\omega}{2}}) \left\{ \omega^4 - 4\omega^2 m_E^2 \left[\frac{1}{4} - 2 \int_{\frac{\omega}{2}}^\omega dq \frac{\omega - q}{2\omega q - \omega^2} \frac{n_q(1 + n_{\omega-q})}{n_{q-\frac{\omega}{2}}^2} \right. \right. \\ & \left. \left. - 2 \int_\omega^\infty dq \frac{q - \omega}{2\omega q - \omega^2} \frac{n_q(1 + n_{q-\omega})}{n_{\frac{\omega}{2}}^2} \right] \right\} + \mathcal{O}(g^8) . \end{aligned} \quad (\text{C.25})$$

In a systematic HTL-computation, we need furthermore to replace the distribution functions through their “classical” limits, $(1+n_q) \rightarrow T/q$ etc.; then the integrals are trivially doable

and we get eq. (5.12). (For notational simplicity we however always display the prefactor $1 + 2n_{\frac{\omega}{2}}$ in its “quantum” form.)

We end by showing that additional terms in $\tilde{G}_\theta^{\text{HTL}}$, on the second row of eq. (C.7), give no ω -dependent contribution in the “naive” limit. At leading order this is clear: both $\Delta_E(Q)$ and $\Delta_T(Q)$ get replaced with Q^2 , so that we are faced with $2g^4 \not\!\!\!\!\!\int_Q 1$, which vanishes in dimensional regularization. At NLO, expanding in Π_E , Π_T and substituting $Q \rightarrow P - Q$, we obtain

$$\begin{aligned} \frac{\tilde{G}_\theta^{\text{HTL}}(P)}{4d_A c_\theta^2} + \frac{\tilde{G}_\chi^{\text{HTL}}(P)}{16d_A c_\chi^2 (D-3)} &= -4g^4 \not\!\!\!\!\!\int_Q \left\{ \frac{\Pi_E(Q)}{Q^2} + (D-2) \frac{\Pi_T(Q)}{Q^2} \right\} \\ &= -4g^4 m_E^2 \not\!\!\!\!\!\int_Q \frac{1}{Q^2}, \end{aligned} \quad (\text{C.26})$$

where we inserted the self-energies from eq. (C.3). This amounts to a “contact contribution”: as discussed around eqs. (5.20), (5.21), at $\mathcal{O}(g^6)$ the two channels do differ in this respect. However, since eq. (C.26) is P -independent, there is no contribution to the spectral function.

C.3. Resummed HTL computation

When eqs. (C.7), (C.8) are kept in their unexpanded forms, the resulting spectral functions can be expressed as convolutions of two “elementary” spectral functions. More precisely, if we make use of the spectral representation,

$$\frac{1}{\Delta_T(Q)} = \int_{-\infty}^{\infty} \frac{d\omega_1}{\pi} \frac{\rho_T(\omega_1, q)}{\omega_1 - iq_n}, \quad \frac{1}{\Delta_T(Q-P)} = \int_{-\infty}^{\infty} \frac{d\omega_2}{\pi} \frac{\rho_T(\omega_2, q)}{\omega_2 - i(q_n - p_n)}, \quad (\text{C.27})$$

then (cf. e.g. ref. [43])

$$\begin{aligned} \text{Im} \left\{ T \sum_{q_n} \frac{1}{\Delta_T(Q) \Delta_T(Q-P)} \right\}_{P \rightarrow (-i[\omega+i0^+], \mathbf{0})} \\ = (1 + 2n_{\frac{\omega}{2}}) \int_{-\infty}^{\infty} \frac{dq^0}{\pi} \rho_T(q^0, q) \rho_T(\omega - q^0, q) \frac{n_{q^0} n_{\omega-q^0}}{n_{\frac{\omega}{2}}^2}. \end{aligned} \quad (\text{C.28})$$

Let us recall a derivation. Inserting eq. (C.27) into the left-hand side of eq. (C.28), the sum can be carried out like in eq. (C.13):

$$\begin{aligned} T \sum_{q_n} \frac{1}{[\omega_1 - iq_n][\omega_2 - i(q_n - p_n)]} &= T \sum_{q_n} T \sum_{r_n} \frac{\beta \delta_{r_n + p_n - q_n}}{(\omega_1 - iq_n)(\omega_2 - ir_n)} \\ &= \int_0^\beta d\tau e^{ip_n \tau} \underbrace{T \sum_{q_n} \frac{e^{-iq_n \tau}}{\omega_1 - iq_n}}_{n_{\omega_1} e^{(\beta-\tau)\omega_1}} \underbrace{T \sum_{r_n} \frac{e^{ir_n \tau}}{\omega_2 - ir_n}}_{n_{\omega_2} e^{\tau\omega_2}} \\ &= n_{\omega_1} n_{\omega_2} \frac{e^{\beta\omega_2} - e^{\beta\omega_1}}{ip_n + \omega_2 - \omega_1}. \end{aligned} \quad (\text{C.29})$$

The summations over q_n and r_n here are “marginally” convergent; their results are discontinuous at $\tau = 0$ and $\tau = \beta$, but there are no Dirac- δ ’s, so the subsequent integration is unproblematic. Setting $p_n \rightarrow -i[\omega + i0^+]$ and taking the imaginary part yields $\text{Im}\{1/[\omega + \omega_2 - \omega_1 + i0^+]\} = -\pi\delta(\omega + \omega_2 - \omega_1)$, so that the integral over ω_1 can be carried out. Afterwards we write

$$\begin{aligned} n_{\omega+\omega_2} n_{\omega_2} (e^{\beta(\omega+\omega_2)} - e^{\beta\omega_2}) &= n_{\omega+\omega_2} (1 + n_{\omega_2}) (e^{\beta\omega} - 1) \\ &= -n_{\omega+\omega_2} n_{-\omega_2} \frac{e^{\frac{\beta\omega}{2}} + 1}{n_{\frac{\omega}{2}}} . \end{aligned} \quad (\text{C.30})$$

Substituting $\omega_2 \rightarrow -q^0$ and making use of $\rho_T(-q^0, q) = -\rho_T(q^0, q)$ we recover eq. (C.28).

In order to make use of eq. (C.28), the objects appearing should be regular enough to allow for a faithful spectral representation; as discussed in sec. 5.4, this requires that the corresponding Euclidean correlators vanish as $p_n \rightarrow \infty$. This is not the case with the terms on the second row of eq. (C.7); however, adding and subtracting Π ’s in the numerator, and shifting $Q \rightarrow P - Q$ where appropriate, it is easy to see that apart from P -independent terms the result can be rewritten as

$$\begin{aligned} \frac{\tilde{G}_\theta^{\text{HTL}}(P)}{4d_A c_\theta^2} &= -2(D-2)g^4 p_n^2 \oint_Q \frac{q^2}{\Delta_T(Q)\Delta_T(Q-P)} \\ &+ 2g^4 \oint_Q \left\{ \frac{\Pi_E(Q)\Pi_E(Q-P)}{\Delta_E(Q)\Delta_E(Q-P)} + (D-2) \frac{\Pi_T(Q)\Pi_T(Q-P)}{\Delta_T(Q)\Delta_T(Q-P)} \right\} + \text{const} . \end{aligned} \quad (\text{C.31})$$

Now all individual “propagators” decrease as a function of the four-momentum.

From eqs. (C.8), (C.27), (C.28), (C.31) we observe that the following spectral functions are needed:

$$\rho_T(q^0, q) \equiv \text{Im} \left\{ \frac{1}{\Delta_T(q_n, q)} \right\}_{q_n \rightarrow -i[q^0 + i0^+]} , \quad (\text{C.32})$$

$$\hat{\rho}_T(q^0, q) \equiv \text{Im} \left\{ \frac{\Pi_T(q_n, q)}{\Delta_T(q_n, q)} \right\}_{q_n \rightarrow -i[q^0 + i0^+]} , \quad (\text{C.33})$$

$$\hat{\rho}_E(q^0, q) \equiv \text{Im} \left\{ \frac{\Pi_E(q_n, q)}{\Delta_E(q_n, q)} \right\}_{q_n \rightarrow -i[q^0 + i0^+]} . \quad (\text{C.34})$$

To determine these we recall that, for $D \rightarrow 4$, the z -integrals in eq. (C.3) can be carried out:

$$\Pi_T(-i[q^0 + i0^+], \mathbf{q}) = \frac{m_E^2}{2} \left\{ \frac{(q^0)^2}{q^2} + \frac{q^0}{2q} \left[1 - \frac{(q^0)^2}{q^2} \right] \ln \frac{q^0 + q + i0^+}{q^0 - q + i0^+} \right\} , \quad (\text{C.35})$$

$$\Pi_E(-i[q^0 + i0^+], \mathbf{q}) = m_E^2 \left[1 - \frac{(q^0)^2}{q^2} \right] \left\{ 1 - \frac{q^0}{2q} \ln \frac{q^0 + q + i0^+}{q^0 - q + i0^+} \right\} . \quad (\text{C.36})$$

Denoting

$$\mathcal{Q} \equiv (q^0, \mathbf{q}) , \quad \mathcal{Q}^2 \equiv (q^0)^2 - q^2 , \quad \eta \equiv \frac{q^0}{q} , \quad (\text{C.37})$$

a straightforward computation leads to

$$\rho_T(\mathcal{Q}) = \begin{cases} \frac{\Gamma_T(\eta)}{\Sigma_T^2(\mathcal{Q}) + \Gamma_T^2(\eta)} , & |\eta| < 1 , \\ \pi \operatorname{sign}(\eta) \delta(\Sigma_T(\mathcal{Q})) , & |\eta| > 1 , \end{cases} \quad (\text{C.38})$$

$$\hat{\rho}_T(\mathcal{Q}) = \begin{cases} \frac{\mathcal{Q}^2 \Gamma_T(\eta)}{\Sigma_T^2(\mathcal{Q}) + \Gamma_T^2(\eta)} , & |\eta| < 1 , \\ \pi \operatorname{sign}(\eta) \mathcal{Q}^2 \delta(\Sigma_T(\mathcal{Q})) , & |\eta| > 1 , \end{cases} \quad (\text{C.39})$$

$$\hat{\rho}_E(\mathcal{Q}) = \begin{cases} \frac{q^2 \Gamma_E(\eta)}{\Sigma_E^2(\mathcal{Q}) + \Gamma_E^2(\eta)} , & |\eta| < 1 , \\ \pi \operatorname{sign}(\eta) q^2 \delta(\Sigma_E(\mathcal{Q})) , & |\eta| > 1 , \end{cases} \quad (\text{C.40})$$

where we have introduced the well-known functions [57]–[59]

$$\Sigma_T(\mathcal{Q}) \equiv -\mathcal{Q}^2 + \frac{m_E^2}{2} \left[\eta^2 + \frac{\eta(1-\eta^2)}{2} \ln \left| \frac{1+\eta}{1-\eta} \right| \right] , \quad (\text{C.41})$$

$$\Gamma_T(\eta) \equiv \frac{\pi m_E^2 \eta (1-\eta^2)}{4} , \quad (\text{C.42})$$

$$\Sigma_E(\mathcal{Q}) \equiv q^2 + m_E^2 \left[1 - \frac{\eta}{2} \ln \left| \frac{1+\eta}{1-\eta} \right| \right] , \quad (\text{C.43})$$

$$\Gamma_E(\eta) \equiv \frac{\pi m_E^2 \eta}{2} . \quad (\text{C.44})$$

For integrals over the poles it is useful to note that

$$\frac{1}{|\partial_{q^0} \Sigma_T(\mathcal{Q})|} \Big|_{\Sigma_T(\mathcal{Q})=0} = \frac{|q^0|(\eta^2 - 1)}{m_E^2 \eta^2 - q^2(\eta^2 - 1)^2} , \quad (\text{C.45})$$

$$\frac{1}{|\partial_q \Sigma_T(\mathcal{Q})|} \Big|_{\Sigma_T(\mathcal{Q})=0} = \frac{q(\eta^2 - 1)}{m_E^2 \eta^2 - 3q^2(\eta^2 - 1)^2} , \quad (\text{C.46})$$

$$\frac{1}{|\partial_{q^0} \Sigma_E(\mathcal{Q})|} \Big|_{\Sigma_E(\mathcal{Q})=0} = \frac{|q^0|(\eta^2 - 1)}{m_E^2 - q^2(\eta^2 - 1)} , \quad (\text{C.47})$$

$$\frac{1}{|\partial_q \Sigma_E(\mathcal{Q})|} \Big|_{\Sigma_E(\mathcal{Q})=0} = \frac{q(\eta^2 - 1)}{m_E^2 - 3q^2(\eta^2 - 1)} . \quad (\text{C.48})$$

The approximate pole positions can also be worked out in specific limits: $\Sigma_T(\mathcal{Q}) = 0$ implies $\eta \approx \frac{m_E}{\sqrt{3}q} + \frac{3\sqrt{3}q}{5m_E}$ for $q \ll m_E$ and $\eta \approx 1 + \frac{m_E^2}{4q^2}$ for $q \gg m_E$, whereas $\Sigma_E(\mathcal{Q}) = 0$ leads to $\eta \approx \frac{m_E}{\sqrt{3}q} + \frac{3\sqrt{3}q}{10m_E}$ for $q \ll m_E$ and $\eta \approx 1 + 2e^{-2(q^2/m_E^2+1)}$ for $q \gg m_E$.

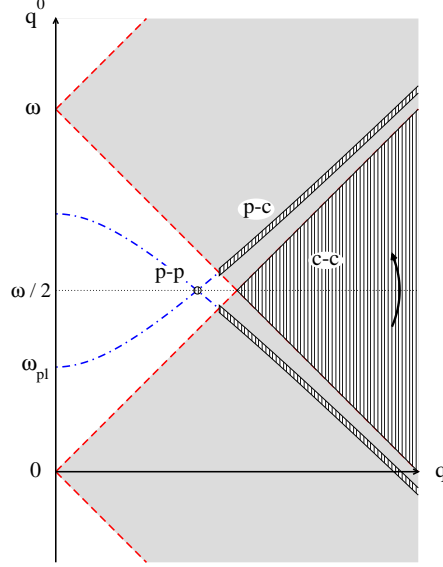


Figure 12: An illustration of the phase space relevant for HTL-resummation. The shaded areas and the dash-dotted blue lines indicate regions in which at least one of the spectral functions has non-zero support; the final contribution emerges from the part where both are simultaneously non-zero, which happens within the hashed areas. For $\omega > 2\omega_{\text{pl}}$, where $\omega_{\text{pl}} \equiv m_{\text{E}}/\sqrt{3}$, this leads to pole-pole (“p-p”), pole-cut (“p-c”) as well as cut-cut (“c-c”) contributions (cf. ref. [43]). As indicated by the arrow, symmetry allows us to restrict the integration to the range $q^0 \geq \omega/2$.

With the spectral functions at hand, we are left to insert them to the imaginary parts of eqs. (C.8), (C.31), rewritten through eq. (C.28):

$$\begin{aligned} \frac{\rho_{\theta}^{\text{HTL}}(\omega)}{4d_A c_{\theta}^2} &= \frac{\pi g^4 (1 + 2n_{\frac{\omega}{2}})}{(4\pi)^2} \left\{ \frac{16}{\pi^2} \int_0^{\infty} dq q^2 \int_{-\infty}^{\infty} dq^0 \right. \\ &\quad \times \left[2 \left(\omega^2 q^2 + \mathcal{Q}^2 ((\omega - q^0)^2 - q^2) \right) \rho_T(q^0, q) \rho_T(\omega - q^0, q) \right. \\ &\quad \left. \left. + \hat{\rho}_E(q^0, q) \hat{\rho}_E(\omega - q^0, q) \right] \frac{n_{q^0} n_{\omega - q^0}}{n_{\frac{\omega}{2}}^2} \right\}, \end{aligned} \quad (\text{C.49})$$

$$\begin{aligned} \frac{-\rho_{\chi}^{\text{HTL}}(\omega)}{16d_A c_{\chi}^2} &= \frac{\pi g^4 (1 + 2n_{\frac{\omega}{2}})}{(4\pi)^2} \left\{ \frac{32\omega^2}{\pi^2} \int_0^{\infty} dq q^4 \int_{-\infty}^{\infty} dq^0 \rho_T(q^0, q) \rho_T(\omega - q^0, q) \frac{n_{q^0} n_{\omega - q^0}}{n_{\frac{\omega}{2}}^2} \right\}, \end{aligned} \quad (\text{C.50})$$

where we have identified the same prefactor as in eq. (C.25). Given the structures of eqs. (C.38)–(C.40), these integrals include three types of contributions, illustrated in fig. 12 (the situation is similar to that in ref. [43]). Taking ρ_{χ}^{HTL} as an example, the “pole–pole” term, which only contributes for $\omega > 2\omega_{\text{pl}}$, where $\omega_{\text{pl}} \equiv m_{\text{E}}/\sqrt{3}$ denotes the “plasmon frequency”,

amounts to

$$\begin{aligned}
& \frac{32\omega^2}{\pi^2} \int_0^\infty dq q^4 \int_0^\infty dq^0 \pi \delta(\Sigma_T(q^0, q)) \pi \delta(\Sigma_T(\omega - q^0, q)) \frac{n_{q^0} n_{\omega - q^0}}{n_{\frac{\omega}{2}}^2} \\
&= 32\omega^2 \int_0^\infty dq q^4 \frac{1}{|\partial_{q^0} \Sigma_T(q^0, q)|} \delta(\Sigma_T(\omega - q^0, q)) \frac{n_{q^0} n_{\omega - q^0}}{n_{\frac{\omega}{2}}^2} \Big|_{\Sigma_T(q^0, q)=0} \\
&= 32\omega^2 q^4 \frac{1}{|\partial_{q^0} \Sigma_T(q^0, q)|} \frac{1}{|-\partial_{q^0} \Sigma_T \frac{dq^0}{dq} + \partial_q \Sigma_T|} \Big|_{\Sigma_T(q^0, q)=0, q^0=\frac{\omega}{2}} \\
&= 16\omega^2 q^4 \frac{1}{|\partial_{q^0} \Sigma_T(\frac{\omega}{2}, q)|} \frac{1}{|\partial_q \Sigma_T(\frac{\omega}{2}, q)|} \Big|_{\Sigma_T(\frac{\omega}{2}, q)=0}. \tag{C.51}
\end{aligned}$$

Here we made use of the fact that along the curve on which $\Sigma_T(\mathcal{Q}) = 0$, $\frac{dq^0}{dq} = -\frac{\partial_q \Sigma_T}{\partial_{q^0} \Sigma_T}$.

For the “pole-cut” contribution, making use of the reflection symmetry with respect to the axis $q^0 = \frac{\omega}{2}$ (cf. fig. 12), we similarly get

$$\begin{aligned}
& \frac{64\omega^2}{\pi^2} \int_0^\infty dq q^4 \int_0^\infty dq^0 \pi \delta(\Sigma_T(q^0, q)) \frac{\Gamma_T(\omega - q^0, q)}{\Sigma_T^2(\omega - q^0, q) + \Gamma_T^2(\omega - q^0, q)} \frac{n_{q^0} n_{\omega - q^0}}{n_{\frac{\omega}{2}}^2} \\
&= \frac{64\omega^2}{\pi} \int_{q_{\min}}^\infty dq q^4 \frac{1}{|\partial_{q^0} \Sigma_T(q^0, q)|} \frac{\Gamma_T(\omega - q^0, q)}{\Sigma_T^2(\omega - q^0, q) + \Gamma_T^2(\omega - q^0, q)} \frac{n_{q^0} n_{\omega - q^0}}{n_{\frac{\omega}{2}}^2} \Big|_{\Sigma_T(q^0, q)=0}, \tag{C.52}
\end{aligned}$$

where the value of q_{\min} is determined from $\Sigma_T(\omega - q_{\min}, q_{\min}) = 0$ for $\omega > \omega_{\text{pl}}$, and $\Sigma_T(\omega + q_{\min}, q_{\min}) = 0$ for $\omega < \omega_{\text{pl}}$. Finally, the “cut-cut” term reads

$$\frac{64\omega^2}{\pi^2} \int_{\frac{\omega}{2}}^\infty dq q^4 \int_{\frac{\omega}{2}}^q dq^0 \frac{\Gamma_T(q^0, q)}{\Sigma_T^2(q^0, q) + \Gamma_T^2(q^0, q)} \frac{\Gamma_T(\omega - q^0, q)}{\Sigma_T^2(\omega - q^0, q) + \Gamma_T^2(\omega - q^0, q)} \frac{n_{q^0} n_{\omega - q^0}}{n_{\frac{\omega}{2}}^2}. \tag{C.53}$$

In a systematic HTL-computation, we are furthermore to replace the Bose distributions in eqs. (C.52), (C.53) through their “classical” limits, $n_{q^0} \rightarrow T/q^0$ etc., so that $n_{q^0} n_{\omega - q^0} / n_{\frac{\omega}{2}}^2 \rightarrow \frac{\omega^2}{4q^0(\omega - q^0)}$. Thereby the HTL-results only feel the scale $\frac{\omega}{m_E}$; this is in analogy with the thermal corrections, which only depend on $\frac{\omega}{\pi T}$.

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